Exercises from Section 1.2.7

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April 27, 2015

1. [*01*] What are H_0, H_1, and H_2?

By definition, we have

$$H_0 = \sum_{1 \le k \le 0} \frac{1}{k} = 0,$$

$$H_1 = \sum_{1 \le k \le 1} \frac{1}{k} = \frac{1}{1} = 1,$$

and

$$H_2 = \sum_{1 \le k \le 2} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} = \frac{3}{2}.$$

2. [*13*] Show that the simple argument used in the text to prove that $H_{2^m} \ge 1 + m/2$ can be slightly modified to prove that $H_{2^m} \le 1 + m$.

We can show that the simple argument used in the text to prove that $H_{2^m} \ge 1 + m/2$ may be slightly modified to prove that $H_{2^m} \le 1 + m$, by noting that for each term, $1/(2^m + k) \le 1/2^m$, as shown in the proof by induction below.

Proposition. $H_{2^m} \le m + 1$.

Proof. Let m be an arbitrary integer such that $m \ge 0$. We must show that $H_{2^m} \le m+1$. In the case that $m = 0$,

$$H_{2^0} = H_1 = 1 \le 0 + 1.$$

Then, assuming

$$H_{2^m} \le m + 1$$

we must show that

$$H_{2^{m+1}} \le m + 2.$$

But

$$\begin{aligned}
H_{2^{m+1}} &= \sum_{1\le k\le 2^{m+1}} \frac{1}{k} \\
&= \sum_{1\le k\le 2^m} \frac{1}{k} + \sum_{2^m+1\le k\le 2^{m+1}} \frac{1}{k} \\
&= H_{2^m} + \sum_{2^m+1\le k\le 2^{m+1}} \frac{1}{k} \\
&= H_{2^m} + \sum_{1\le k\le 2^m} \frac{1}{2^m+k} \\
&\le H_{2^m} + \sum_{1\le k\le 2^m} \frac{1}{2^m} \\
&= H_{2^m} + \frac{2^m}{2^m} \\
&= H_{2^m} + 1 \\
&\le m+1+1 \\
&= m+2
\end{aligned}$$

as we needed to show. □

3. [*M21*] Generalize the argument used in the previous exercise to show that, for $r > 1$, the sum $H_n^{(r)}$ remains bounded for all n. Find an upper bound.

Proposition. $H_n^{(r)} \le \frac{2^{r-1}}{2^{r-1}-1}$ *for* $r > 1$.

Proof. Let n be an arbitrary nonnegative integer and r an arbitrary real such that $r > 1$. We must show that

$$H_n^{(r)} \le \frac{2^{r-1}}{2^{r-1}-1}.$$

First note that for arbitrary $m \ge 1$, we may show that

$$\sum_{1\le k\le 2^{m-1}} \frac{1}{k^r} \le \sum_{0\le k<m} \frac{2^k}{2^{kr}}.$$

If $m = 1$,

$$\begin{aligned}
\sum_{1\le k\le 2^{1-1}} \frac{1}{k^r} &= \sum_{1\le k\le 0} \frac{1}{k^r} \\
&= 0 \\
&\le 1 \\
&= \frac{2^0}{2^{(0)r}} \\
&= \sum_{0\le k<1} \frac{2^k}{2^{kr}}.
\end{aligned}$$

Then assuming

$$\sum_{1\le k\le 2^{m-1}} \frac{1}{k^r} \le \sum_{0\le k<m} \frac{2^k}{2^{kr}}.$$

we must show that

$$\sum_{1\le k\le 2^m} \frac{1}{k^r} \le \sum_{0\le k<m+1} \frac{2^k}{2^{kr}}.$$

But

$$\begin{aligned}
\sum_{1\le k\le 2^m} \frac{1}{k^r} &= \sum_{1\le k\le 2^{m-1}} \frac{1}{k^r} + \sum_{2^{m-1}+1\le k\le 2^m} \frac{1}{k^r} \\
&\le \sum_{0\le k<m} \frac{2^k}{2^{kr}} + \sum_{2^{m-1}+1\le k\le 2^m} \frac{1}{k^r} \\
&= \sum_{0\le k<m} \frac{2^k}{2^{kr}} + \sum_{1\le k\le 2^{m-1}} \frac{1}{(2^{m-1}+k)^r} \\
&\le \sum_{0\le k<m} \frac{2^k}{2^{kr}} + \sum_{1\le k\le 2^{m-1}} \frac{1}{(2^{m-1})^r} \\
&= \sum_{0\le k<m} \frac{2^k}{2^{kr}} + \frac{2^{m-1}}{(2^{m-1})^r} \\
&= \sum_{0\le k<m} \frac{2^k}{2^{kr}} + \frac{2^{m-1}}{2^{(m-1)r}} \\
&\le \sum_{0\le k<m} \frac{2^k}{2^{kr}} + \frac{2^m}{2^{mr}} \\
&= \sum_{0\le k<m+1} \frac{2^k}{2^{kr}}
\end{aligned}$$

and hence the noted inequality. We now continue with the main proof.

Since $2^{r-1} > 1$, we have both in the case that $n = 0$ that

$$H_0^{(r)} = \sum_{1\le k\le 0} \frac{1}{k^r} = 0 \le \frac{2^{r-1}}{2^{r-1}-1}$$

and in the case that $n = 1 = 2^{m-1}$ for $m = 1$ that

$$H_1^{(r)} = \sum_{1\le k\le 1} \frac{1}{k^r} = 1 \le \frac{2^{r-1}}{2^{r-1}-1}.$$

Then, for arbitrary $m \geq 1$, and since $2^{-mr+m+r-1} = \frac{2^{r-1}}{2^{(r-1)m}} < 1 < 2^{r-1}$,

$$
\begin{aligned}
H^{(r)}_{2^m-1} &= \sum_{1 \leq k \leq 2^m-1} \frac{1}{k^r} \\
&\leq \sum_{0 \leq k < m} \frac{2^k}{2^{kr}} \\
&= \sum_{0 \leq k < m} \frac{1}{2^{(r-1)k}} \\
&= \sum_{0 \leq k \leq m-1} 2^{(-r+1)k} \\
&= \frac{2^{(-r+1)0} - 2^{(-r+1)m}}{1 - 2^{-r+1}} \\
&= \frac{1 - 2^{(-r+1)m}}{1 - 2^{-r+1}} \\
&= \frac{(2^{m(r-1)} - 1)/2^{m(r-1)}}{(2^{r-1} - 1)/2^{r-1}} \\
&= \frac{2^{r-1}(2^{m(r-1)} - 1)}{2^{m(r-1)}(2^{r-1} - 1)} \\
&= \frac{2^{m(r-1)+(r-1)} - 2^{r-1}}{2^{m(r-1)+(r-1)} - 2^{m(r-1)}} \\
&= \frac{2^{-m(r-1)}(2^{m(r-1)+(r-1)} - 2^{r-1})}{2^{r-1} - 1} \\
&= \frac{2^{-m(r-1)}2^{m(r-1)+(r-1)} - 2^{-m(r-1)}2^{r-1}}{2^{r-1} - 1} \\
&= \frac{2^{r-1} - 2^{-m(r-1)+r-1}}{2^{r-1} - 1} \\
&= \frac{2^{r-1} - 2^{-mr+m+r-1}}{2^{r-1} - 1} \\
&\leq \frac{2^{r-1}}{2^{r-1} - 1}
\end{aligned}
$$

as we needed to show. □

▶ **4.** [*10*] Decide which of the following statements are true for all positive integers n: (a) $H_n < \ln n$. (b) $H_n > \ln n$. (c) $H_n > \ln n + \gamma$.

In summary, (a) is false, while (b) and (c) are true, the justification for each enumerated below.

a) $H_n < \ln n$ is not true for all positive integers n, as may be seen by considering $n = 1$, in which case, $H_1 = 1 \not< 0 = \ln 1$.

b) $H_n > \ln n$ is true for all positive integers n, as may be deduced from Eq. (3), since $\gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \epsilon > 0$.

c) $H_n > \ln n + \gamma$ is true for all positive integers n, as may also be deduced from Eq. (3), since $\frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \epsilon > 0$.

5. [*15*] Give the value of H_{10000} to 15 decimal places, using the tables in Appendix A.

From Eq. (3) we know

$$H_{10000} = \ln 10000 + \gamma + \frac{1}{2(10000)} - \frac{1}{12(10000)^2} + \frac{1}{120(10000)^4} - \epsilon$$

for $0 < \epsilon < \frac{1}{252(10000)^6}$. Letting $\epsilon' = \frac{1}{120(10000)^4} - \epsilon > 0$, since

$$\begin{aligned}\epsilon' &< \frac{1}{120(10000)^4}\\ &= \frac{1}{1.2 \times 10^{18}}\\ &< \frac{1}{10^{18}},\end{aligned}$$

we may ignore ϵ' in order to approximate H_{10000} to only 15 decimal places as

$$\begin{aligned}H_{10000} &\approx \ln 10000 + \gamma + \frac{1}{2(10,000)} - \frac{1}{12(10,000)^2}\\ &= 4\ln 10 + \gamma + \frac{59999}{1200000000}.\end{aligned}$$

Given

$$\begin{aligned}\ln 10 &= 2.30258\ 50929\ 94045\ 6+\\ \gamma &= 0.57721\ 56649\ 01532\ 8+\\ \frac{59999}{1200000000} &= 0.00004\ 99991\ 66666\ 6+\end{aligned}$$

we may compute the sum as

$$\begin{array}{rl} & 2.30258\ 50929\ 94045\ 6\\ & 2.30258\ 50929\ 94045\ 6\\ & 2.30258\ 50929\ 94045\ 6\\ & 2.30258\ 50929\ 94045\ 6\\ & 0.57721\ 56649\ 01532\ 8\\ + & 0.00004\ 99991\ 66666\ 6\\ \hline & 9.78760\ 60360\ 44381\ 8\end{array}$$

That is,

$$H_{10000} \approx 9.78760\ 60360\ 44382\ \ldots$$

6. [*M15*] Prove that the harmonic numbers are directly related to Stirling's numbers, which were introduced in the previous section; in fact,

$$H_n = \left[{n+1 \atop 2}\right] \Big/ n!.$$

Proposition. $H_n = \left[{n+1 \atop 2}\right]/n!$.

Proof. Let n be an arbitrary nonnegative integer. We must show that

$$H_n = \left[{n+1 \atop 2}\right] \Big/ n!.$$

In the case that $n = 0$,

$$\begin{aligned} H_0 &= \sum_{1 \le k \le 0} \frac{1}{k} \\ &= 0 \\ &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 0+1 \\ 2 \end{bmatrix} \Big/ 0!. \end{aligned}$$

Then, assuming

$$H_n = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} \Big/ n!$$

we must show that

$$H_{n+1} = \begin{bmatrix} n+2 \\ 2 \end{bmatrix} \Big/ (n+1)!.$$

But

$$\begin{aligned} H_{n+1} &= H_n + \frac{1}{n+1} \\ &= \left(\begin{bmatrix} n+1 \\ 2 \end{bmatrix} \Big/ n! \right) + \frac{1}{n+1} \\ &= \left((n+1) \begin{bmatrix} n+1 \\ 2 \end{bmatrix} + n! \right) \Big/ (n+1)! \\ &= \left((n+1) \begin{bmatrix} n+1 \\ 2 \end{bmatrix} + \begin{bmatrix} n+1 \\ 1 \end{bmatrix} \right) \Big/ (n+1)! && \text{from Eq. (50)} \\ &= \left((n+1) \begin{bmatrix} n+1 \\ 2 \end{bmatrix} + \begin{bmatrix} n+1 \\ 2-1 \end{bmatrix} \right) \Big/ (n+1)! \\ &= \begin{bmatrix} n+2 \\ 2 \end{bmatrix} \Big/ (n+1)! && \text{from Eq. (46)} \end{aligned}$$

as we needed to show. □

7. [*M21*] Let $T(m,n) = H_m + H_n - H_{mn}$. (a) Show that when m or n increases, $T(m,n)$ never increases (assuming that m and n are positive). (b) Compute the minimum and maximum values of $T(m,n)$ for $m, n > 0$.

We may provide a proof and determine bounds.

a) We may show that $T(m,n)$ never increases.

Proposition. *$T(m+1,n) \le T(m,n)$ for m,n positive integers.*

Proof. Define $T(m,n)$ as

$$T(m,n) = H_m + H_n - H_{mn}$$

and let m and n be arbitrary positive integers. We must show that

$$T(m+1,n) - T(m,n) \le 0.$$

But

$$\begin{aligned}
T(m+1,n) - T(m,n) &= \big(H_{m+1} + H_n - H_{(m+1)n}\big) - (H_m + H_n - H_{mn}) \\
&= H_{m+1} + H_n - H_{(m+1)n} - H_m - H_n + H_{mn} \\
&= H_{m+1} - H_{(m+1)n} - H_m + H_{mn} \\
&= \frac{1}{m+1} - \sum_{mn+1 \le k \le mn+n} \frac{1}{k} \\
&\le \frac{1}{m+1} - \sum_{mn+1 \le k \le mn+n} \frac{1}{mn+n} \\
&= \frac{1}{m+1} - \frac{n}{mn+n} \\
&= \frac{1}{m+1} - \frac{1}{m+1} \\
&= 0
\end{aligned}$$

as we needed to show. □

b) We may determine both the lower and upper bounds of $T(m,n)$, for m, n positive integers. Since $T(m,n)$ never increases, we know that the lower bound corresponds to the limit as $m \to \infty$, and from Eq. (3),

$$\lim_{m\to\infty} T(m,n) = \lim_{m\to\infty} (H_m + H_n - H_{mn}) = \lim_{m\to\infty} (H_m - \ln m) = \gamma.$$

Similarly, since $T(m,n)$ never increases, we know that the upper bound corresponds to $m = n = 1$, and

$$T(1,1) = H_1 + H_1 - H_1 = H_1 = 1.$$

[*AMM* **70** (1963), 575–577]

8. [*HM18*] Compare Eq. (8) with $\sum_{k=1}^{n} \ln k$; estimate the difference as a function of n.

Given Eq. (8)

$$\sum_{1\le k\le n} H_k = (n+1)H_n - n$$

we may estimate the difference with $\sum_{1\le k\le n} \ln k$. First, we note from Eq. (3) that

$$\begin{aligned}
\sum_{1\le k\le n} H_k &= (n+1)H_n - n \\
&\approx (n+1)\,(\ln n + \gamma + 1/2n) - n \\
&= (n+1)\ln n + (n+1)\gamma + (n+1)/2n - n \\
&= (n+1)\ln n - n + (n+1)\gamma + (n+1)/2n \\
&\approx (n+1)\ln n - n + (n+1)\gamma + 1/2 \\
&= (n+1)\ln n - n + n\gamma + \gamma + 1/2 \\
&= (n+1)\ln n - n(1-\gamma) + (\gamma + 1/2).
\end{aligned}$$

Second, we note from Stirling's approximation that

$$\begin{aligned}\sum_{1\le k\le n} \ln k &= \ln n! \\ &\approx \ln \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \\ &= \ln\sqrt{2\pi} + \frac{1}{2}\ln n + n\ln n - n\ln e \\ &= \ln\sqrt{2\pi} + \left(n+\frac{1}{2}\right)\ln n - n \\ &= \left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi}.\end{aligned}$$

And so,

$$\begin{aligned}&\sum_{1\le k\le n} H_k - \sum_{1\le k\le n} \ln k \\ &\approx \left((n+1)\ln n - n(1-\gamma) + \left(\gamma + \frac{1}{2}\right)\right) - \left(\left(n+\frac{1}{2}\right)\ln n - n + \ln\sqrt{2\pi}\right) \\ &= (n+1)\ln n + -n + \gamma n + \gamma + \frac{1}{2} - \left(n+\frac{1}{2}\right)\ln n + n - \ln\sqrt{2\pi} \\ &= \gamma n + \left(n+1-n-\frac{1}{2}\right)\ln n + \gamma + \frac{1}{2} - \ln\sqrt{2\pi} \\ &= \gamma n + \frac{1}{2}\ln n + \gamma + \frac{1}{2} - \ln\sqrt{2\pi} \\ &\approx \gamma n + \frac{1}{2}\ln n + .158.\end{aligned}$$

▶ **9.** [*M18*] Theorem A applies only when $x > 0$; what is the value of the sum considered when $x = -1$?

We make a proposition and offer proof in the case that $x = -1$.

Proposition. $\sum_{1\le k\le n}\binom{n}{k}(-1)^k H_k = -\frac{1}{n}$.

Proof. Let n be an arbitrary positive integer. We must show that

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^k H_k = -\frac{1}{n}.$$

If $n = 1$,

$$\sum_{1\le k\le 1}\binom{1}{k}(-1)^k H_k = \binom{1}{1}(-1)^1 H_1 = -\frac{1}{1}.$$

Then, assuming

$$\sum_{1\le k\le n}\binom{n}{k}(-1)^k H_k = -\frac{1}{n},$$

we must show that

$$\sum_{1\le k\le n+1}\binom{n+1}{k}(-1)^k H_k = -\frac{1}{n+1}.$$

But

$$\begin{aligned}
&\sum_{1\le k\le n+1}\binom{n+1}{k}(-1)^k H_k\\
&= \sum_{1\le k\le n+1}\left(\binom{n}{k}+\binom{n}{k-1}\right)(-1)^k H_k\\
&= \sum_{1\le k\le n+1}\binom{n}{k}(-1)^k H_k + \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^k H_k\\
&= \sum_{1\le k\le n}\binom{n}{k}(-1)^k H_k + \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^k H_k\\
&= -\frac{1}{n} + \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^k H_k\\
&= -\frac{1}{n} - \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1} H_k\\
&= -\frac{1}{n} - \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1}\left(H_{k-1}+\frac{1}{k}\right)\\
&= -\frac{1}{n} - \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1} H_{k-1} - \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1}\frac{1}{k}\\
&= -\frac{1}{n} - \sum_{0\le k\le n}\binom{n}{k}(-1)^k H_k - \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1}\frac{1}{k}\\
&= -\frac{1}{n} - \sum_{1\le k\le n}\binom{n}{k}(-1)^k H_k - \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1}\frac{1}{k}\\
&= -\frac{1}{n} + \frac{1}{n} - \sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1}\frac{1}{k}\\
&= -\sum_{1\le k\le n+1}\binom{n}{k-1}(-1)^{k-1}\frac{1}{k}\\
&= -\sum_{1\le k\le n+1}\frac{1}{n+1}\binom{n+1}{k}(-1)^{k-1} \qquad \text{from Eq. 1.2.6-(7)}\\
&= \frac{1}{n+1}\sum_{1\le k\le n+1}\binom{n+1}{k}(-1)^k\\
&= \frac{1}{n+1}\left(-\binom{n+1}{0}(-1)^0 + \sum_{0\le k\le n+1}\binom{n+1}{k}(-1)^k\right)\\
&= \frac{1}{n+1}\left(-1+(1-1)^{n+1}\right)\\
&= -\frac{1}{n+1}
\end{aligned}$$

as we needed to show. □

10. [*M20*] (*Summation by parts.*) We have used special cases of the general method of summation by parts in exercise 1.2.4-42 and in the derivation of Eq. (9). Prove the general formula

$$\sum_{1\le k<n}(a_{k+1}-a_k)b_k = a_n b_n - a_1 b_1 - \sum_{1\le k<n} a_{k+1}(b_{k+1}-b_k).$$

Proposition. $\sum_{1\le k<n}(a_{k+1}-a_k)b_k = a_nb_n - a_1b_1 - \sum_{1\le k<n} a_{k+1}(b_{k+1}-b_k)$.

Proof. Let n be an arbitrary positive integer. We must show that

$$\sum_{1\le k<n}(a_{k+1}-a_k)b_k = a_nb_n - a_1b_1 - \sum_{1\le k<n} a_{k+1}(b_{k+1}-b_k).$$

But

$$\begin{aligned}
&\sum_{1\le k<n}(a_{k+1}-a_k)b_k \\
&\quad= \sum_{1\le k<n} a_{k+1}b_k - \sum_{1\le k<n} a_kb_k \\
&\quad= \sum_{1\le k<n} a_{k+1}b_k - \sum_{0\le k<n-1} a_{k+1}b_{k+1} \\
&\quad= \sum_{1\le k<n} a_{k+1}b_k - \left(a_1b_1 + \sum_{1\le k<n} a_{k+1}b_{k+1} - a_nb_n\right) \\
&\quad= a_nb_n - a_1b_1 + \sum_{1\le k<n} a_{k+1}b_k - \sum_{1\le k<n} a_{k+1}b_{k+1} \\
&\quad= a_nb_n - a_1b_1 - \left(\sum_{1\le k<n} a_{k+1}b_{k+1} - \sum_{1\le k<n} a_{k+1}b_k\right) \\
&\quad= a_nb_n - a_1b_1 - \sum_{1\le k<n} a_{k+1}(b_{k+1}-b_k)
\end{aligned}$$

as we needed to show. □

▶ **11.** [*M21*] Using summation by parts, evaluate

$$\sum_{1<k\le n} \frac{1}{k(k-1)} H_k.$$

The sum may be evaluated using summation by parts as

$$
\begin{aligned}
&\sum_{1<k\le n} \frac{1}{k(k-1)} H_k \\
&= \sum_{1<k\le n} \frac{k-(k-1)}{k(k-1)} H_k \\
&= \sum_{1<k\le n} \left(\frac{1}{k-1} - \frac{1}{k}\right) H_k \\
&= \sum_{1<k\le n} \left(-\frac{1}{(k+1)-1} - -\frac{1}{k-1}\right) H_k \\
&= \sum_{1\le k<n} \left(-\frac{1}{k+1} - -\frac{1}{k}\right) H_{k+1} \\
&= -\frac{1}{n} H_{n+1} - -\frac{1}{1} H_{1+1} - \sum_{1\le k<n} -\frac{1}{k+1} \left(H_{(k+1)+1} - H_{k+1}\right) \\
&= -\frac{1}{n}\left(H_n + \frac{1}{n+1}\right) + 1 + \frac{1}{2} + \sum_{1\le k<n} \frac{1}{k+1} \left(H_{(k+1)+1} - H_{k+1}\right) \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{1\le k<n} \frac{1}{k+1} \left(H_{(k+1)+1} - H_{k+1}\right) \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{1\le k<n} \frac{1}{k+1}\frac{1}{k+2} \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{1\le k<n} \frac{1}{(k+1)(k+2)} \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{1\le k<n} \frac{(k+2)-(k+1)}{(k+1)(k+2)} \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{1\le k<n} \frac{1}{k+1} - \frac{1}{k+2} \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{1\le k<n} \frac{1}{k+1} - \sum_{1\le k<n} \frac{1}{k+2} \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{2\le k\le n} \frac{1}{k} - \sum_{3\le k\le n+1} \frac{1}{k} \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + \sum_{1\le k\le n} \frac{1}{k} - 1 - \left(\sum_{1\le k\le n} \frac{1}{k} - 1 - \frac{1}{2} + \frac{1}{n+1}\right) \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + H_n - 1 - \left(H_n - 1 - \frac{1}{2} + \frac{1}{n+1}\right) \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 1 + \frac{1}{2} + H_n - 1 - H_n + 1 + \frac{1}{2} - \frac{1}{n+1} \\
&= -\frac{1}{n} H_n - \frac{1}{n}\frac{1}{n+1} + 2 - \frac{1}{n+1} \\
&= 2 - H_n/n - \left(\frac{1}{n(n+1)} + \frac{1}{n+1}\right) \\
&= 2 - H_n/n - \frac{n+1}{n(n+1)} \\
&= 2 - H_n/n - 1/n.
\end{aligned}
$$

▶ **12.** [*M10*] Evaluate H_∞^{1000} correct to at least 100 decimal places.

By definition

$$H_\infty^{1000} = \sum_{k\geq 1} \frac{1}{k^{1000}} = 1 + \sum_{k\geq 2} \frac{1}{k^{1000}} = 1 + \epsilon$$

where $\epsilon \leq \frac{2^{1000-1}}{2^{1000-1}-1} - 1$ from exercise 3, and

$$\begin{aligned} \epsilon &\leq \frac{2^{1000-1}}{2^{1000-1}-1} - 1 \\ &= \frac{2^{999}}{2^{999}-1} - 1 \\ &= \frac{2^{999}-1+1}{2^{999}-1} - 1 \\ &= \frac{1}{2^{999}-1} + 1 - 1 \\ &= \frac{1}{2^{999}-1} \\ &< \frac{1}{2^{998}} \\ &= \frac{1}{10^{998\ln 2/\ln 10}} \qquad < \frac{1}{10^{300}} \end{aligned}$$

so that

$$H_\infty^{1000} = 1.000\ldots$$

to at least 100 decimal places.

13. [*M22*] Prove the identity

$$\sum_{k=1}^{n} \frac{x^k}{k} = H_n + \sum_{k=1}^{n} \binom{n}{k} \frac{(x-1)^k}{k}.$$

(Note in particular the special case $x = 0$, which gives us an identity related to exercise 1.2.6-48.)

Proposition. $\sum_{1\leq k\leq n} \frac{x^k}{k} = H_n + \sum_{1\leq k\leq n} \binom{n}{k} \frac{(x-1)^k}{k}$.

Proof. Let n be an arbitrary positive integer and x an arbitrary real. We must show that

$$\sum_{1\leq k\leq n} \frac{x^k}{k} = H_n + \sum_{1\leq k\leq n} \binom{n}{k} \frac{(x-1)^k}{k}.$$

In the case that $n = 1$

$$\sum_{1\leq k\leq 1} \frac{x^k}{k} = x = 1 + \binom{1}{1}(x-1) = H_1 + \sum_{1\leq k\leq 1} \binom{1}{k} \frac{(x-1)^k}{k}.$$

Then, assuming

$$\sum_{1\leq k\leq n} \frac{x^k}{k} = H_n + \sum_{1\leq k\leq n} \binom{n}{k} \frac{(x-1)^k}{k}$$

we must show that

$$\sum_{1\leq k\leq n+1} \frac{x^k}{k} = H_{n+1} + \sum_{1\leq k\leq n+1} \binom{n+1}{k} \frac{(x-1)^k}{k}.$$

But

$$\begin{aligned}
&\sum_{1\le k\le n+1}\frac{x^k}{k}\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{x^{n+1}}{n+1}\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\frac{x^{n+1}}{n+1}-\frac{1}{n+1}\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\frac{1}{n+1}\left((1+(x-1))^{n+1}-1\right)\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\frac{1}{n+1}\left(\sum_{0\le k\le n+1}\binom{n+1}{k}(x-1)^k-1\right)\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\frac{1}{n+1}\left(\sum_{0\le k\le n+1}\binom{n+1}{k}(x-1)^k-\binom{n+1}{0}(x-1)^0\right)\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\frac{1}{n+1}\sum_{1\le k\le n+1}\binom{n+1}{k}(x-1)^k\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\frac{1}{n+1}\sum_{1\le k\le n+1}\frac{n+1}{k}\binom{n}{k-1}(x-1)^k \qquad \text{from Eq. 1.2.6-(7)}\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\sum_{1\le k\le n+1}\frac{n+1}{n+1}\frac{1}{k}\binom{n}{k-1}(x-1)^k\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\sum_{1\le k\le n+1}\binom{n}{k-1}\frac{(x-1)^k}{k}\\
&=\sum_{1\le k\le n}\frac{x^k}{k}+\frac{1}{n+1}+\binom{n}{n+1}\frac{(x-1)^{n+1}}{n+1}+\sum_{1\le k\le n+1}\binom{n}{k-1}\frac{(x-1)^k}{k}\\
&=H_n+\sum_{1\le k\le n}\binom{n}{k}\frac{(x-1)^k}{k}+\frac{1}{n+1}+\binom{n}{n+1}\frac{(x-1)^{n+1}}{n+1}+\sum_{1\le k\le n+1}\binom{n}{k-1}\frac{(x-1)^k}{k}\\
&=H_{n+1}+\sum_{1\le k\le n+1}\binom{n}{k}\frac{(x-1)^k}{k}+\sum_{1\le k\le n+1}\binom{n}{k-1}\frac{(x-1)^k}{k}\\
&=H_{n+1}+\sum_{1\le k\le n+1}\left(\binom{n}{k}+\binom{n}{k-1}\right)\frac{(x-1)^k}{k}\\
&=H_{n+1}+\sum_{1\le k\le n+1}\binom{n+1}{k}\frac{(x-1)^k}{k}
\end{aligned}$$

as we needed to show. □

14. [*M22*] Show that $\sum_{k=1}^{n} H_k/k = \frac{1}{2}(H_n^2 + H_n^{(2)})$, and evaluate $\sum_{k=1}^{n} H_k/(k+1)$.

We may prove the identity.

Proposition. $\sum_{1\le k\le n} H_k/k = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right)$.

Proof. Let n be an arbitrary nonnegative integer. We must show that

$$\sum_{1 \le k \le n} H_k/k = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right).$$

But

$$\begin{aligned}
&\sum_{1 \le k \le n} H_k/k \\
&\sum_{1 \le k \le n} \frac{1}{k} H_k \\
&\quad = \sum_{1 \le k \le n} \frac{1}{k} \sum_{1 \le j \le k} \frac{1}{j} \\
&\quad = \sum_{1 \le k \le n} \sum_{1 \le j \le k} \frac{1}{k}\frac{1}{j} \\
&\quad = \frac{1}{2}\left(\left(\sum_{1 \le k \le n} \frac{1}{k}\right)^2 + \left(\sum_{1 \le k \le n} \frac{1}{k^2}\right)\right) && \text{from Eq. 1.2.3-(13)} \\
&\quad = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right)
\end{aligned}$$

as we needed to show. □

Thus, we may evaluate the sum as

$$
\begin{aligned}
&\sum_{1\le k\le n} H_k/(k+1)\\
&= \sum_{1\le k\le n} \frac{1}{k+1} H_k\\
&= \sum_{1\le k\le n} \frac{1}{k+1} \sum_{1\le j\le k} \frac{1}{j}\\
&= \sum_{1\le k\le n} \sum_{1\le j\le k} \frac{1}{k+1}\frac{1}{j}\\
&= \sum_{2\le k\le n+1} \sum_{1\le j\le k-1} \frac{1}{k}\frac{1}{j}\\
&= \sum_{2\le k\le n+1} \frac{1}{k}\left(-\frac{1}{k} + \sum_{1\le j\le k}\frac{1}{j}\right)\\
&= -\sum_{2\le k\le n+1} \frac{1}{k}\frac{1}{k} + \sum_{2\le k\le n+1}\frac{1}{k}\sum_{1\le j\le k}\frac{1}{j}\\
&= -\sum_{2\le k\le n+1} \frac{1}{k^2} + \sum_{2\le k\le n+1}\frac{1}{k}\sum_{1\le j\le k}\frac{1}{j}\\
&= -\left(-1 + \sum_{1\le k\le n+1}\frac{1}{k^2}\right) + \sum_{2\le k\le n+1}\frac{1}{k}\sum_{1\le j\le k}\frac{1}{j}\\
&= -\left(-1 + H_{n+1}^{(2)}\right) + \sum_{2\le k\le n+1}\frac{1}{k}\sum_{1\le j\le k}\frac{1}{j}\\
&= 1 - H_{n+1}^{(2)} + \sum_{2\le k\le n+1}\frac{1}{k}\sum_{1\le j\le k}\frac{1}{j}\\
&= 1 - H_{n+1}^{(2)} + \sum_{1\le k\le n+1}\frac{1}{k}\sum_{1\le j\le k}\frac{1}{j} - \frac{1}{1}\sum_{1\le j\le 1}\frac{1}{j}\\
&= 1 - H_{n+1}^{(2)} + \sum_{1\le k\le n+1}\frac{1}{k}\sum_{1\le j\le k}\frac{1}{j} - 1\\
&= 1 - H_{n+1}^{(2)} + \sum_{1\le k\le n+1}\frac{1}{k}H_k - 1\\
&= -H_{n+1}^{(2)} + \frac{1}{2}\left(H_{n+1}^2 + H_{n+1}^{(2)}\right)\\
&= -H_{n+1}^{(2)} + \frac{1}{2}H_{n+1}^2 + \frac{1}{2}H_{n+1}^{(2)}\\
&= -H_{n+1}^{(2)} + \frac{1}{2}H_{n+1}^2 + \frac{1}{2}H_{n+1}^{(2)}\\
&= \frac{1}{2}H_{n+1}^2 - \frac{1}{2}H_{n+1}^{(2)}\\
&= \frac{1}{2}\left(H_{n+1}^2 - H_{n+1}^{(2)}\right).
\end{aligned}
$$

▶ **15.** [*M23*] Express $\sum_{k=1}^{n} H_k^2$ in terms of n and H_n.

The sum is

$$
\begin{aligned}
&\sum_{1\le k\le n} H_k^2\\
&= \sum_{1\le k\le n} H_k H_k\\
&= \sum_{1\le k\le n} H_k \sum_{1\le j\le k} \frac{1}{j}\\
&= \sum_{1\le k\le n} \sum_{1\le j\le k} H_k \frac{1}{j}\\
&= \sum_{1\le j\le n} \sum_{j\le k\le n} H_k \frac{1}{j}\\
&= \sum_{1\le j\le n} \frac{1}{j} \sum_{j\le k\le n} H_k\\
&= \sum_{1\le j\le n} \frac{1}{j} \left(\sum_{1\le k\le n} H_k - \sum_{1\le k\le j-1} H_k \right)\\
&= \sum_{1\le j\le n} \frac{1}{j} \left(((n+1)H_n - n) - ((j-1+1)H_{j-1} - (j-1)) \right) \qquad \text{from Eq. (8)}\\
&= \sum_{1\le j\le n} \frac{1}{j} \left((n+1)H_n - n - jH_{j-1} + j - 1 \right)\\
&= ((n+1)H_n - n - 1) \sum_{1\le j\le n} \frac{1}{j} - \sum_{1\le j\le n} \frac{1}{j} jH_{j-1} + \sum_{1\le j\le n} \frac{1}{j} j\\
&= ((n+1)H_n - n - 1)\, H_n - \sum_{1\le j\le n} H_{j-1} + \sum_{1\le j\le n} 1\\
&= (n+1)H_n^2 - nH_n - H_n - \left(\sum_{1\le j\le n} H_j - H_n \right) + n\\
&= (n+1)H_n^2 - nH_n - H_n - ((n+1)H_n - n - H_n) + n\\
&= (n+1)H_n^2 - nH_n - H_n - (n+1)H_n + n + H_n + n\\
&= (n+1)H_n^2 - nH_n - (n+1)H_n + 2n\\
&= (n+1)H_n^2 - (2n+1)H_n + 2n.
\end{aligned}
$$

16. [*18*] Express the sum $1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$ in terms of harmonic numbers.

The sum of all n unit fractions with odd denominators through $2n-1$ may be expressed as

$$\begin{aligned}
\sum_{1\le k\le n} \frac{1}{2k-1} &= \sum_{\substack{1\le k\le 2n-1\\ \text{k odd}}} \frac{1}{k} \\
&= \sum_{1\le k\le 2n-1} \frac{1}{k} - \sum_{\substack{1\le k\le 2n-1\\ \text{k even}}} \frac{1}{k} \\
&= \sum_{1\le k\le 2n-1} \frac{1}{k} - \sum_{1\le k\le n-1} \frac{1}{2k} \\
&= H_{2n-1} - \frac{1}{2} \sum_{1\le k\le n-1} \frac{1}{k} \\
&= H_{2n-1} - \frac{1}{2} H_{n-1}.
\end{aligned}$$

17. [*M24*] (E. Waring, 1782.) Let p be an odd prime. Show that the numerator of H_{p-1} is divisible by p.

Proposition. *If p is an odd prime, the numerator of H_{p-1} is divisible by p.*

Proof. Let p be an arbitrary odd prime. We must show that the numerator of H_{p-1} is divisible by p. That is, that

$$(p-1)!H_{p-1} = \sum_{1\le k\le p-1} \frac{(p-1)!}{k} \equiv 0 \pmod{p}.$$

From exercise 1.2.4-19, the *law of inverses*, we may find a k' such that

$$kk' \equiv 1 \pmod{p}$$

since $k \perp p$. Note that $1 \le k' \le p-1$ and that each k' is unique such that $\{k | 1 \le k \le p-1\} = \{k' | kk' \equiv 1 \pmod{p}\}$. Also note that since p is an odd prime by hypothesis, $-\frac{(p-1)}{2}$ is an integer. Then, from Wilson's theorem

$$(p-1)! \equiv -1 \pmod{p}$$

we have that

$$\begin{aligned}
\sum_{1\le k\le p-1} \frac{(p-1)!}{k} &\equiv - \sum_{1\le k\le p-1} \frac{1}{k} \\
&\equiv - \sum_{1\le k\le p-1} \frac{kk'}{k} \\
&\equiv - \sum_{1\le k\le p-1} k' \\
&\equiv - \sum_{1\le k\le p-1} k \\
&\equiv - \frac{p(p-1)}{2} \\
&\equiv 0 \pmod{p}
\end{aligned}$$

as we needed to show. ☐

[Hardy and Wright, *An Introduction to the Theory of Numbers*, Section 7.8]

18. [*M33*] (J. Selfridge.) What is the highest power of 2 that divides the numerator of $1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$?

We want to find the highest power of 2 that divides the numerator of

$$\sum_{1 \le k \le n} \frac{1}{2k-1},$$

assuming n positive.

Let m be the integer such that $n = 2^r m$ for some integer r. We know that m exists and is odd, as it is the product of the odd primes from the prime factorization of n.

We then have

$$\begin{aligned}\sum_{1 \le k \le n} \frac{1}{2k-1} &= \sum_{1 \le k \le 2^r m} \frac{1}{2k-1} \\ &= \sum_{0 \le j \le m-1} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1},\end{aligned}$$

which we may prove by induction on m.

If $m = 1$,

$$\sum_{1 \le k \le 2^r} \frac{1}{2k-1} = \sum_{1 \le k \le 2^r} \frac{1}{(0)2^{r+1} + 2k - 1} = \sum_{0 \le j \le 0} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1}.$$

Then, assuming

$$\sum_{1 \le k \le 2^r m} \frac{1}{2k-1} = \sum_{0 \le j \le m-1} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1},$$

we must show that

$$\sum_{1 \le k \le 2^r (m+1)} \frac{1}{2k-1} = \sum_{0 \le j \le m} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1}.$$

But

$$\begin{aligned}\sum_{1 \le k \le 2^r (m+1)} \frac{1}{2k-1} &= \sum_{1 \le k \le 2^r m} \frac{1}{2k-1} + \sum_{2^r m + 1 \le k \le 2^r (m+1)} \frac{1}{2k-1} \\ &= \sum_{0 \le j \le m-1} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1} + \sum_{2^r m + 1 \le k \le 2^r (m+1)} \frac{1}{2k-1} \\ &= \sum_{0 \le j \le m-1} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1} + \sum_{1 \le k - 2^r m \le 2^r} \frac{1}{2k-1} \\ &= \sum_{0 \le j \le m-1} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1} + \sum_{1 \le k \le 2^r} \frac{1}{2(k + 2^r m) - 1} \\ &= \sum_{0 \le j \le m-1} \sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1} + \sum_{1 \le k \le 2^r} \frac{1}{m2^{r+1} m + 2k - 1} \\ &= \sum_{1 \le k \le 2^r m} \frac{1}{2k-1}\end{aligned}$$

and hence the identity.

Let

$$P_j = \prod_{1 \le k \le 2^r} j2^{r+1} + 2k - 1$$

be the common denominator such that

$$\sum_{1 \le k \le 2^r} \frac{1}{j2^{r+1} + 2k - 1} = \sum_{1 \le k \le 2^r} \frac{P_j}{P_j(j2^{r+1} + 2k - 1)} = \sum_{1 \le k \le 2^r} \frac{P_j/(j2^{r+1} + 2k - 1)}{P_j}.$$

That is, such that the numerator of $\sum_{1 \le k \le n} \frac{1}{2k-1}$ is

$$\sum_{0 \le j \le m-1} \sum_{1 \le k \le 2^r} \frac{P_j}{j2^{r+1} + 2k - 1},$$

m sets of 2^r terms, each of the form P_j over a distinct odd residue of 2^{r+1}. Each ratio itself is an integer and a distinct odd residue of 2^{r+1}, and the sum of 2^r distinct odd residues is $2^{r^2} m_j = 2^{2r} m_j$ for some integer m_j by the odd number theorem, m_j odd. That is, the numerator of $\sum_{1 \le k \le n} \frac{1}{2k-1}$ is

$$\sum_{0 \le j \le m-1} 2^{2r} m_j = 2^{2r} \sum_{0 \le j \le m-1} m_j.$$

Since m is odd, we know the sum of m odd terms m_j is itself an odd number. Let this be M, so that the numerator of $\sum_{1 \le k \le n} \frac{1}{2k-1}$ is

$$2^{2r} M$$

for some odd integer M. That is, 2^{2r} is the highest power of 2 that divides the numerator of

$$\sum_{1 \le k \le n} \frac{1}{2k-1}$$

where m is the odd integer such that $n = 2^r m$ for some integer r.

[*AMM* **67** (1960), 924–925]

► **19.** [*M30*] List all nonnegative integers n for which H_n is an integer. [*Hint:* If H_n has odd numerator and even denominator, it cannot be an integer.]

The nonnegative integers n for which H_n is an integer are $n = 0$ and $n = 1$, since $H_0 = 0$ and $H_1 = 1$. To see why these are the *only* n, consider the following. Let $k = \lfloor \lg n \rfloor$ with $n \ge 2$, so that $2^k \le n < 2^{k+1}$ and $k \ge 1$, and let

$$P = \prod_{1 \le i \le n} 2^k m$$

be the common denominator for each term of H_n, m odd but P even. We know that m exists and is odd, as it is the product of the odd primes from a prime factorization of the common denominator. Then

$$\begin{aligned} H_n &= \sum_{1 \le j \le n} \frac{1}{j} \\ &= \sum_{1 \le j \le n} \frac{P}{Pj} \\ &= \sum_{1 \le j \le n} \frac{P/j}{P} \\ &= \sum_{1 \le j \le n} P/j \Big/ P. \end{aligned}$$

If H_n is an integer, then so is $H_n - 1$. But

$$H_n - 1 = \sum_{2 \le j \le n} P/j \Big/ P = \frac{M}{P}$$

for $M = \sum_{2 \le j \le n} P/j$. Each term in M is even except for the term with $j = 2^k$, $P/2^k = m$, which means M is odd. But the divisor P is even. This means their ratio cannot possibly be an integer, and hence the claim.

20. [*HM22*] There is an analytic way to approach summation problems such as the one leading to Theorem A in this section: If $f(x) = \sum_{k \ge 0} a_k x^k$, and this series converges for $x = x_0$, prove that

$$\sum_{k \ge 0} a_k x_0^k H_k = \int_0^1 \frac{f(x_0) - f(x_0 y)}{1 - y} dy.$$

Proposition. *If* $f(x) = \sum_{k \ge 0} a_k x^k$ *and* $f(x)$ *converges for* $x = x_0$ *then* $\sum_{k \ge 0} a_k x_0^k H_k = \int_0^1 \frac{f(x_0) - f(x_0 y)}{1 - y} dy$.

Proof. Let $f(x) = \sum_{k \ge 0} a_k x^k$ be a series with arbitrary coefficients a_k such that $f(x)$ converges for $x = x_0$. We must show that

$$\sum_{k \ge 0} a_k x_0^k H_k = \int_0^1 \frac{f(x_0) - f(x_0 y)}{1 - y} dy.$$

But

$$\begin{aligned}
\sum_{k \ge 0} a_k x_0^k H_k &= \sum_{k \ge 0} a_k x_0^k \sum_{1 \le j \le k} \frac{1}{j} \\
&= \sum_{k \ge 0} a_k x_0^k \sum_{1 \le j \le k} \int_0^1 y^{j-1} dy \\
&= \sum_{k \ge 0} a_k x_0^k \int_0^1 \sum_{1 \le j \le k} y^{j-1} dy \\
&= \sum_{k \ge 0} a_k x_0^k \int_0^1 \sum_{0 \le j \le k-1} y^j dy \\
&= \sum_{k \ge 0} a_k x_0^k \int_0^1 \frac{y^0 - y^{k-1+1}}{1 - y} dy \\
&= \sum_{k \ge 0} a_k x_0^k \int_0^1 \frac{1 - y^k}{1 - y} dy \\
&= \int_0^1 \frac{1}{1 - y} \sum_{k \ge 0} (a_k x_0^k - a_k x_0^k y^k) dy \\
&= \int_0^1 \frac{1}{1 - y} \left(\sum_{k \ge 0} a_k x_0^k - \sum_{k \ge 0} a_k (x_0 y)^k \right) dy \\
&= \int_0^1 \frac{1}{1 - y} \left(f(x_0) - f(x_0 y) \right) dy \\
&= \int_0^1 \frac{f(x_0) - f(x_0 y)}{1 - y} dy
\end{aligned}$$

as we needed to show. □

[*AMM* **69** (1962), 239; H. W. Gould, *Mathematics Magazine* **34** (1961), 317–321]

21. [*M24*] Evaluate $\sum_{k=1}^{n} H_k/(n+1-k)$.

The difference between the sum for n and $n+1$ is given as

$$
\begin{aligned}
&\sum_{1\le k\le n+1} \frac{H_k}{n+2-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= \frac{H_{n+1}}{n+2-(n+1)} + \sum_{1\le k\le n} \frac{H_k}{n+2-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= H_{n+1} + \sum_{1\le k\le n} \frac{H_k}{n+2-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= H_{n+1} + \sum_{1\le k\le n} \left(\frac{H_k}{n+2-k} - \frac{H_k}{n+1-k}\right) \\
&= H_{n+1} + \sum_{1\le k\le n} \frac{H_k}{n+2-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= H_{n+1} + \sum_{0\le k\le n-1} \frac{H_{k+1}}{n+1-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= H_{n+1} + \frac{H_1}{n+1} - H_{n+1} + \sum_{1\le k\le n} \frac{H_{k+1}}{n+1-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= \frac{1}{n+1} + \sum_{1\le k\le n} \frac{H_{k+1}}{n+1-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= \frac{1}{n+1} + \sum_{1\le k\le n} \frac{H_k + \frac{1}{k+1}}{n+1-k} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= \frac{1}{n+1} + \sum_{1\le k\le n} \frac{H_k}{n+1-k} + \sum_{1\le k\le n} \frac{1}{(k+1)(n+1-k)} - \sum_{1\le k\le n} \frac{H_k}{n+1-k} \\
&= \frac{1}{n+1} + \sum_{1\le k\le n} \frac{1}{(k+1)(n+1-k)} \\
&= \frac{1}{n+1} + \sum_{1\le k\le n} \left(\frac{1}{(n+2)(k+1)} + \frac{1}{(n+2)(n+1-k)}\right) \\
&= \frac{1}{n+1} + \frac{1}{n+2} \sum_{1\le k\le n} \left(\frac{1}{k+1} + \frac{1}{n+1-k}\right) \\
&= \frac{1}{n+1} + \frac{1}{n+2} \sum_{1\le k\le n} \frac{1}{k+1} + \frac{1}{n+2} \sum_{1\le k\le n} \frac{1}{n+1-k} \\
&= \frac{1}{n+1} + \frac{1}{n+2} \sum_{2\le k\le n+1} \frac{1}{k} + \frac{1}{n+2} \sum_{1\le k\le n} \frac{1}{k} \\
&= \frac{1}{n+1} + \frac{1}{n+2} \left(-\frac{1}{1} + \frac{1}{n+1} + \sum_{1\le k\le n} \frac{1}{k}\right) + \frac{H_n}{n+2} \\
&= \frac{1}{n+1} + \frac{1}{n+2} \left(-\frac{1}{1} + \frac{1}{n+1} + H_n\right) + \frac{H_n}{n+2} \\
&= \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{H_n - 1}{n+2} + \frac{H_n}{n+2} \\
&= \frac{1}{n+1} + \frac{H_n}{n+2} - \frac{1}{n+2} + \frac{H_{n+1}}{n+2} \\
&= \frac{1}{(n+1)(n+2)} + \frac{H_n}{n+2} + \frac{H_{n+1}}{n+2} \\
&= \frac{H_{n+1}}{n+2} + \frac{H_{n+1}}{n+2} \\
&= 2\frac{H_{n+1}}{n+2}.
\end{aligned}
$$

Then, in the case that $n = 1$

$$\sum_{1\le k\le 1} \frac{H_k}{1+1-k} = \frac{H_k}{1+1-k} = H_1 = 1 = \frac{9}{4} - \frac{5}{4} = \left(1+\frac{1}{2}\right)^2 - \left(1+\frac{1}{2^2}\right) = H_2^2 - H_2^{(2)}$$

and assuming

$$\sum_{1\le k\le n} \frac{H_k}{n+1-k} = H_{n+1}^2 - H_{n+1}^{(2)}$$

it may be shown that

$$\sum_{1\le k\le n+1} \frac{H_k}{n+2-k} = H_{n+2}^2 - H_{n+2}^{(2)}$$

as

$$\begin{aligned}
&\sum_{1\le k\le n+1} \frac{H_k}{n+2-k} \\
&= \sum_{1\le k\le n} \frac{H_k}{n+1-k} + 2\frac{H_{n+1}}{n+2} \\
&= H_{n+1}^2 - H_{n+1}^{(2)} + 2\frac{H_{n+1}}{n+2} \\
&= H_{n+1}^2 + 2\frac{H_{n+1}}{n+2} + \frac{1}{(n+2)^2} - H_{n+1}^{(2)} - \frac{1}{(n+2)^2} \\
&= \left(H_{n+1} + \frac{1}{n+2}\right)^2 - \left(H_{n+1}^{(2)} + \frac{1}{(n+2)^2}\right) \\
&= H_{n+2}^2 - H_{n+2}^{(2)}.
\end{aligned}$$

That is

$$\sum_{1\le k\le n} \frac{H_k}{n+1-k} = H_{n+1}^2 - H_{n+1}^{(2)}.$$

22. [*M28*] Evaluate $\sum_{k=0}^{n} H_k H_{n-k}$.

From summation by parts and exercise 21,

$$\begin{aligned}
&\sum_{0\le k\le n} H_k H_{n-k} \\
&= \sum_{1\le k\le n} H_k H_{n-k} \\
&= \left(\sum_{1\le j\le n} H_j\right) H_{n-n} - \sum_{1\le k\le n}\left(\sum_{1\le j\le k} H_j\right)\left(H_{n-(k+1)} - H_{n-k}\right) \\
&= \left((n+1)H_n - n\right) H_0 + \sum_{1\le k\le n-1} \left((k+1)H_k - k\right)\frac{1}{n-k} \\
&= \sum_{1\le k\le n-1} \frac{(k+1)H_k - k}{n-k} \\
&= \sum_{1\le k\le n-1} \frac{(n-k+1)H_{n-k} - n + k}{k} \\
&= \sum_{1\le k\le n-1} \frac{nH_{n-k} - kH_{n-k} + H_{n-k} - n + k}{k} \\
&= \sum_{1\le k\le n-1} \frac{nH_{n-k}}{k} - \sum_{1\le k\le n-1} \frac{kH_{n-k}}{k} + \sum_{1\le k\le n-1} \frac{H_{n-k}}{k} - \sum_{1\le k\le n-1} \frac{n}{k} + \sum_{1\le k\le n-1} \frac{k}{k} \\
&= n\sum_{1\le k\le n-1} \frac{H_{n-k}}{k} - \sum_{1\le k\le n-1} H_{n-k} + \sum_{1\le k\le n-1} \frac{H_{n-k}}{k} - n\sum_{1\le k\le n-1} \frac{1}{k} + \sum_{1\le k\le n-1} 1 \\
&= n\sum_{1\le k\le n-1} \frac{H_{n-k}}{k} - \sum_{1\le k\le n-1} H_{n-k} + \sum_{1\le k\le n-1} \frac{H_{n-k}}{k} - nH_{n-1} + n - 1 \\
&= n\sum_{1\le k\le n-1} \frac{H_k}{n-k} - \sum_{1\le k\le n-1} H_k + \sum_{1\le k\le n-1} \frac{H_k}{n-k} - n(H_n - 1) \\
&= (n+1)\sum_{1\le k\le n-1} \frac{H_k}{n-k} - \left(((n-1)+1)H_{n-1} - (n-1)\right) - n(H_n - 1) \\
&= (n+1)\sum_{1\le k\le n-1} \frac{H_k}{n-k} - nH_{n-1} + n - 1 - n(H_n - 1) \\
&= (n+1)\sum_{1\le k\le n-1} \frac{H_k}{n-k} - n(H_n - 1) - n(H_n - 1) \\
&= (n+1)\sum_{1\le k\le n-1} \frac{H_k}{n-k} - 2n(H_n - 1) \\
&= (n+1)\left(H_n^2 - H_n^{(2)}\right) - 2n(H_n - 1).
\end{aligned}$$

▶ **23.** [*HM20*] By considering the function $\Gamma'(x)/\Gamma(x)$, show how we can get a natural generalization of H_n to noninteger values of n. You may use the fact that $\Gamma'(1) = -\gamma$, anticipating the next exercise.

We can get a natural generalization of H_n to noninteger values of n by considering the function $\Gamma'(x)/\Gamma(x)$, using the fact that $\Gamma'(1) = -\gamma$.

By definition,

$$\Gamma(x+1) = x\Gamma(x),$$

and so

$$\begin{aligned}\Gamma'(x+1) &= (x\Gamma(x))' \\ &= x'\Gamma(x) + x\Gamma'(x) \\ &= \Gamma(x) + x\Gamma'(x)\end{aligned}$$

if and only if

$$\begin{aligned}\frac{\Gamma'(x+1)}{\Gamma(x+1)} &= \frac{\Gamma(x) + x\Gamma'(x)}{\Gamma(x+1)} \\ &= \frac{\Gamma(x) + x\Gamma'(x)}{x\Gamma(x)} \\ &= \frac{\Gamma(x)}{x\Gamma(x)} + \frac{x\Gamma'(x)}{x\Gamma(x)} \\ &= \frac{1}{x} + \frac{\Gamma'(x)}{\Gamma(x)},\end{aligned}$$

giving us a natural generalization of H_n to noninteger values of n as

$$H_x = \frac{\Gamma'(x+1)}{\Gamma(x+1)} + \gamma.$$

Note that in the case that $x = 0$

$$H_0 = \frac{\Gamma'(1)}{\Gamma(1)} + \gamma = \frac{-\gamma}{1} + \gamma = 0,$$

and assuming

$$H_x = \frac{\Gamma'(x+1)}{\Gamma(x+1)} + \gamma$$

we have that

$$\begin{aligned}H_{x+1} &= H_x + \frac{1}{x+1} \\ &= \frac{\Gamma'(x+1)}{\Gamma(x+1)} + \gamma + \frac{1}{x+1} \\ &= \frac{\Gamma'(x+1)}{\Gamma(x+1)} + \frac{1}{x+1}\gamma \\ &= \frac{\Gamma'(x+2)}{\Gamma(x+2)} + \gamma,\end{aligned}$$

proving the identity holds for all nonnegative integers x.

24. [*HM21*] Show that

$$xe^{\gamma x} \prod_{k\geq 1} \left(\left(1 + \frac{x}{k}\right) e^{-x/k}\right) = \frac{1}{\Gamma(x)}.$$

(Consider the partial products of this infinite product.)

Proposition. $xe^{\gamma x} \prod_{k\geq 1} \left(\left(1 + \frac{x}{k}\right) e^{-x/k}\right) = \frac{1}{\Gamma(x)}$.

Proof. Let x be an arbitrary real. We must show that

$$xe^{\gamma x} \prod_{k \geq 1} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) = \frac{1}{\Gamma(x)}.$$

But since

$$\gamma = \lim_{n \to \infty} (H_n - \ln n)$$

we have that

$$\begin{aligned}
&xe^{\gamma x} \prod_{k \geq 1} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) \\
&= xe^{\gamma x} \lim_{n \to \infty} \prod_{1 \leq k \leq n} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) \\
&= \lim_{n \to \infty} xe^{\gamma x} \prod_{1 \leq k \leq n} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) \\
&= \lim_{n \to \infty} xe^{(H_n - \ln n)x} \prod_{1 \leq k \leq n} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) \\
&= \lim_{n \to \infty} x\frac{e^{xH_n}}{e^{x \ln n}} \prod_{1 \leq k \leq n} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) \\
&= \lim_{n \to \infty} x\frac{e^{xH_n}}{n^x} \prod_{1 \leq k \leq n} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) \\
&= \lim_{n \to \infty} \frac{x}{n^x} e^{xH_n} \prod_{1 \leq k \leq n} \left(\left(1 + \frac{x}{k}\right) e^{-x/k} \right) \\
&= \lim_{n \to \infty} \frac{x}{n^x} e^{xH_n} \left(\prod_{1 \leq k \leq n} \left(1 + \frac{x}{k}\right) \right) \left(\prod_{1 \leq k \leq n} e^{-x/k} \right) \\
&= \lim_{n \to \infty} \frac{x}{n^x} e^{xH_n} \left(\prod_{1 \leq k \leq n} \left(1 + \frac{x}{k}\right) \right) e^{-x \sum_{1 \leq k \leq n} 1/k} \\
&= \lim_{n \to \infty} \frac{x}{n^x} e^{xH_n} \left(\prod_{1 \leq k \leq n} \left(1 + \frac{x}{k}\right) \right) e^{-xH_n} \\
&= \lim_{n \to \infty} \frac{x}{n^x} \prod_{1 \leq k \leq n} \left(1 + \frac{x}{k}\right) \\
&= \lim_{n \to \infty} \frac{x}{n^x} \prod_{1 \leq k \leq n} \frac{x + k}{k} \\
&= \lim_{n \to \infty} \frac{x}{n^x} \frac{\prod_{1 \leq k \leq n}(x + k)}{\prod_{1 \leq k \leq n} k} \\
&= \lim_{n \to \infty} \frac{x}{n^x} \frac{\prod_{1 \leq k \leq n}(x + k)}{n!} \\
&= \lim_{n \to \infty} \frac{x \prod_{1 \leq k \leq n}(x + k)}{n^x n!} \\
&= \frac{1}{\Gamma(x)}
\end{aligned}$$

as we needed to show. ☐