Exercises from Section 1.2.6

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1. [*00*] How many combinations of n things taken $n-1$ at a time are possible?

There are $\binom{n}{n-1} = n$ combinations of n things taken $n-1$ at a time. Intuitively, each distinct set of $n-1$ objects leaves out a single item, and there are n items.

2. [*00*] What is $\binom{0}{0}$?

We have

$$\begin{aligned}\binom{0}{0} &= \frac{0!}{0!(0-0)!}\\ &= \frac{1}{1}\\ &= 1.\end{aligned}$$

Intuitively, there is only a single way to choose nothing from nothing.

3. [*00*] How many bridge hands (13 cards out of a 52-card deck) are possible?

There are $\binom{52}{13} = 635013559600$ possible bridge hands, as we are choosing 13 from 52 things.

4. [*10*] Give the answer to exercise 3 as a product of prime numbers.

The answer to exercise 3 was $\binom{52}{13} = \frac{52!}{13!(52-13)!} = \frac{52!}{13!39!}$. We can use Eq. 1.2.5-(8) to determine the prime factorization of each factorial and then the answer as a whole as

$$\begin{aligned}\binom{52}{13} &= \frac{52!}{13!39!}\\ &= \frac{2^{49}\cdot 3^{23}\cdot 5^{12}\cdot 7^{8}\cdot 11^{4}\cdot 13^{4}\cdot 17^{3}\cdot 19^{2}\cdot 23^{2}\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47}{2^{10}\cdot 3^{5}\cdot 5^{2}\cdot 7\cdot 11\cdot 13\cdot 2^{35}\cdot 3^{18}\cdot 5^{8}\cdot 7^{5}\cdot 11^{3}\cdot 13^{3}\cdot 17^{2}\cdot 19^{2}\cdot 23\cdot 29\cdot 31\cdot 37}\\ &= \frac{2^{49}\cdot 3^{23}\cdot 5^{12}\cdot 7^{8}\cdot 11^{4}\cdot 13^{4}\cdot 17^{3}\cdot 19^{2}\cdot 23^{2}\cdot 29\cdot 31\cdot 37\cdot 41\cdot 43\cdot 47}{2^{45}\cdot 3^{23}\cdot 5^{10}\cdot 7^{6}\cdot 11^{4}\cdot 13^{4}\cdot 17^{2}\cdot 19^{2}\cdot 23\cdot 29\cdot 31\cdot 37}\\ &= 2^{4}\cdot 5^{2}\cdot 7^{2}\cdot 17\cdot 23\cdot 41\cdot 43\cdot 47.\end{aligned}$$

▶ **5.** [*05*] Use Pascal's triangle to explain the fact that $11^4 = 14641$.

By the binomial theorem,

$$\begin{aligned}
11^4 &= (10+1)^4 \\
&= \sum_{0\le k\le 4} \binom{4}{k} 10^k 1^{4-k} \\
&= \binom{4}{4}10^4 + \binom{4}{3}10^3 + \binom{4}{2}10^2 + \binom{4}{1}10^1 + \binom{4}{0}10^0 \\
&= (1)10^4 + (4)10^3 + (6)10^2 + (4)10^1 + (1)10^0 \\
&= 14641.
\end{aligned}$$

That is, the digits represent the row in Pascal's triangle for $\binom{4}{k}$, $0 \le k \le 4$.

▶ **6.** [*10*] Pascal's triangle (Table 1) can be extended in all directions by use of the addition formula, Eq. (9). Find the three rows that go on *top* of Table 1 (i.e., for $r = -1$, -2, and -3).

Using Eq. (9)

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

we can extend Pascal's triangle (Table 1) for $r = -1$, -2, and -3 as

r	$\binom{r}{0}$	$\binom{r}{1}$	$\binom{r}{2}$	$\binom{r}{3}$	$\binom{r}{4}$	$\binom{r}{5}$	$\binom{r}{6}$	$\binom{r}{7}$	$\binom{r}{8}$	$\binom{r}{9}$
-3	1	-3	6	-10	15	-21	28	-36	45	-55
-2	1	-2	3	-4	5	-6	7	-8	9	-10
-1	1	-1	1	-1	1	-1	1	-1	1	-1

since $\binom{r}{0} = 1$, $\binom{r}{1} = r$, and $\binom{r-1}{k} = \binom{r}{k} - \binom{r-1}{k-1}$.

7. [*12*] If n is a fixed positive integer, what value of k makes $\binom{n}{k}$ a maximum?

Proposition. *$\binom{n}{k} \le \binom{n}{\lceil n/2\rceil} = \binom{n}{\lfloor n/2\rfloor}$ for all integers $n \ge 1$, k.*

Proof. Let n, k be arbitrary integers such that $n \ge 1$. We must show that

$$\binom{n}{k} \le \binom{n}{\lceil n/2\rceil} = \binom{n}{\lfloor n/2\rfloor}.$$

First, we must show that the binomial coefficient is monotone in k, $0 \le k \le \lceil \frac{n}{2} \rceil$. That is, that

$$\binom{n}{k-1} \le \binom{n}{k} \qquad 1 \le k \le \left\lceil \frac{n}{2} \right\rceil.$$

But

$$
\begin{aligned}
k \le \left\lceil \frac{n}{2} \right\rceil \quad &\Longleftrightarrow \quad k \le \frac{n+1}{2} \\
&\Longleftrightarrow \quad 2k \le n+1 \\
&\Longleftrightarrow \quad k \le n-k+1 \\
&\Longleftrightarrow \quad k \le n-(k-1) \\
&\Longleftrightarrow \quad \frac{k}{n-(k-1)} \le 1 \\
&\Longleftrightarrow \quad \frac{k}{n-(k-1)} \binom{n}{k} \le \binom{n}{k} \\
&\Longleftrightarrow \quad \frac{k}{n-(k-1)} \frac{n!}{k!(n-k)!} \le \binom{n}{k} \\
&\Longleftrightarrow \quad \frac{n!}{(k-1)!(n-(k-1))!} \le \binom{n}{k} \\
&\Longleftrightarrow \quad \binom{n}{k-1} \le \binom{n}{k}.
\end{aligned}
$$

And by definition, since $\binom{n}{k} = 0 < 1 = \binom{n}{0}$ for $k < 0$, we have in general that if $k \le \lceil \frac{n}{2} \rceil$

$$\binom{n}{k-1} \le \binom{n}{k},$$

or equivalently that if $k \le \lceil \frac{n}{2} \rceil$

$$\binom{n}{k} \le \binom{n}{\lceil n/2 \rceil}.$$

In the case that $k > \lceil \frac{n}{2} \rceil$,

$$
\begin{aligned}
k > \left\lceil \frac{n}{2} \right\rceil \quad &\Longleftrightarrow \quad -k < -\left\lceil \frac{n}{2} \right\rceil \\
&\Longleftrightarrow \quad n-k < n - \left\lceil \frac{n}{2} \right\rceil \\
&\Longleftrightarrow \quad n-k < n + \left\lfloor \frac{-n}{2} \right\rfloor \\
&\Longleftrightarrow \quad n-k < \left\lfloor n + \frac{-n}{2} \right\rfloor \\
&\Longleftrightarrow \quad n-k < \left\lfloor \frac{n}{2} \right\rfloor
\end{aligned}
$$

so that

$$\binom{n}{k} = \binom{n}{n-k} \le \binom{n}{\lfloor n/2 \rfloor} = \binom{n}{\lceil n/2 \rceil}.$$

That is for all integers $n \ge 1$ and k

$$\binom{n}{k} \le \binom{n}{\lceil n/2 \rceil} = \binom{n}{\lfloor n/2 \rfloor}$$

as we needed to show. ☐

8. [*00*] What property of Pascal's triangle is reflected in the "symmetry condition," Eq. (6)?

The property of Pascal's triangle that is reflected in the "symmetry condition," Eq. (6), is the symmetry of the triangle itself. That is, each row, not counting zeros, is palindromic: values read the same left to right and vice versa.

9. [*01*] What is the value of $\binom{n}{n}$? (Consider all integers n.)

Since $\binom{n}{n} = \binom{n}{0} = 1$ and $\binom{n}{k} = 0$ for $k < 0$, we have

$$\binom{n}{n} = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

▶ **10.** [*M25*] If p is prime, show that:

a) $\displaystyle\binom{n}{p} \equiv \left\lfloor \frac{n}{p} \right\rfloor$ (modulo p).

b) $\displaystyle\binom{p}{k} \equiv 0$ (modulo p), for $1 \leq k \leq p-1$.

c) $\displaystyle\binom{p-1}{k} \equiv (-1)^k$ (modulo p), for $0 \leq k \leq p-1$.

d) $\displaystyle\binom{p+1}{k} \equiv 0$ (modulo p), for $2 \leq k \leq p-1$.

e) (É. Lucas, 1877.)
$$\binom{n}{k} \equiv \binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor}\binom{n \bmod p}{k \bmod p} \quad \text{(modulo } p\text{)}.$$

f) If the p-ary number system representations of n and k are

$$\begin{aligned} n &= a_r p^r + \cdots + a_1 p + a_0, \\ k &= b_r p^r + \cdots + b_1 p + b_0, \end{aligned} \qquad \text{then} \qquad \binom{n}{k} \equiv \binom{a_r}{b_r} \cdots \binom{a_1}{b_1}\binom{a_0}{b_0} \quad \text{(modulo } p\text{)}.$$

The answers to exercise 10 follow below.

a) We may prove the equivalence.

Proposition. $\binom{n}{p} \equiv \left\lfloor \frac{n}{p} \right\rfloor \pmod{p}$.

Proof. Let n and p be arbitrary integers such that $n \geq 1$ and p prime. We must show that
$$\binom{n}{p} \equiv \left\lfloor \frac{n}{p} \right\rfloor \pmod{p}.$$
But given (e) with $k = p$,
$$\begin{aligned} \binom{n}{p} &\equiv \binom{\lfloor n/p \rfloor}{\lfloor p/p \rfloor}\binom{n \bmod p}{p \bmod p} \\ &\equiv \binom{\lfloor n/p \rfloor}{1}\binom{n \bmod p}{0} \\ &\equiv \binom{\lfloor n/p \rfloor}{1} \\ &\equiv \left\lfloor \frac{n}{p} \right\rfloor \pmod{p} \end{aligned}$$

as we needed to show. □

b) We may prove the equivalence.

Proposition. $\binom{p}{k} \equiv 0 \pmod{p}$ *for* $1 \leq k \leq p-1$.

Proof. Let p and k be arbitrary integers such that p is prime and $1 \le k \le p-1$. We must show that

$$\binom{p}{k} \equiv 0 \pmod{p}.$$

But given (e) with $n = p$, and since $k/p < 1$,

$$\begin{aligned}\binom{p}{k} &\equiv \binom{\lfloor p/p \rfloor}{\lfloor k/p \rfloor}\binom{p \bmod p}{k \bmod p} \\ &\equiv \binom{1}{0}\binom{0}{k} \\ &\equiv 1 \cdot 0 \\ &\equiv 0 \pmod{p}\end{aligned}$$

as we needed to show. □

c) We may prove the equivalence.

Proposition. $\binom{p-1}{k} \equiv (-1)^k \pmod{p}$ *for* $0 \le k \le p-1$.

Proof. Let p and k be arbitrary integers such that p is prime and $0 \le k \le p-1$. We must show that

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p}.$$

If $k = 0$, then clearly

$$\binom{p-1}{0} \equiv 1 \equiv (-1)^k \pmod{p}.$$

Then, assuming

$$\binom{p-1}{k} \equiv (-1)^k \pmod{p},$$

we must show that

$$\binom{p-1}{k+1} \equiv (-1)^{k+1} \pmod{p}.$$

But by the addition formula and (b),

$$\begin{aligned}\binom{p-1}{k+1} &\equiv \binom{p}{k+1} - \binom{p-1}{k} \\ &\equiv \binom{p}{k+1} - (-1)^k \\ &\equiv 0 - (-1)^k \\ &\equiv (-1)^{k+1} \pmod{p}\end{aligned}$$

as we needed to show. □

d) We may prove the equivalence.

Proposition. $\binom{p+1}{k} \equiv 0 \pmod{p}$ *for* $2 \le k \le p-1$.

Proof. Let p and k be arbitrary integers such that p is prime and $2 \le k \le p-1$. We must show that

$$\binom{p+1}{k} \equiv 0 \pmod{p}.$$

But by the addition formula Eq. (3) and (b),

$$\begin{aligned}\binom{p+1}{k} &\equiv \binom{p}{k} + \binom{p}{k-1} \\ &\equiv 0 + 0 \\ &\equiv 0 \pmod{p}\end{aligned}$$

as we needed to show. □

e) We may prove the equivalence.

Proposition. $\binom{n}{k} \equiv \binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor}\binom{n \bmod p}{k \bmod p} \pmod{p}$.

Proof. Let n, k, and p be arbitrary integers such that p is prime. We must show that

$$\binom{n}{k} \equiv \binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor}\binom{n \bmod p}{k \bmod p} \pmod{p}.$$

Note that

$$\begin{aligned} n &= \left\lfloor \frac{n}{p} \right\rfloor p + (n \bmod p) & 0 \le n \bmod p < p, \\ k &= \left\lfloor \frac{k}{p} \right\rfloor p + (k \bmod p) & 0 \le k \bmod p < p. \end{aligned}$$

Also note from Eq. (7), that

$$s\binom{r}{s} = r\binom{r-1}{s-1},$$

with $r = p^{\lfloor n/p \rfloor}$ and $s = k$ implies

$$\binom{p^{\lfloor n/p \rfloor}}{s} \equiv 0 \pmod{p}$$

and for arbitrary x that

$$(x+1)^{\lfloor n/p \rfloor p} \equiv (x^p + 1)^{\lfloor n/p \rfloor} \pmod{p}.$$

Then, for arbitrary x by the binomial theorem,

$$\begin{aligned}\sum_{0 \le k \le n} \binom{n}{k} x^k &= (x+1)^n \\ &= (x+1)^{\lfloor n/p \rfloor p + (n \bmod p)} \\ &= (x+1)^{\lfloor n/p \rfloor p}(x+1)^{n \bmod p} \\ &\equiv (x^p+1)^{\lfloor n/p \rfloor}(x+1)^{n \bmod p} \pmod{p} \\ &= \left(\sum_{0 \le i \le \lfloor n/p \rfloor} \binom{\lfloor n/p \rfloor}{i} x^{ip}\right)\left(\sum_{0 \le j \le n \bmod p} \binom{n \bmod p}{j} x^j\right) \\ &= \sum_{0 \le ip+j \le \lfloor n/p \rfloor p + (n \bmod p)} \binom{\lfloor n/p \rfloor}{i}\binom{n \bmod p}{j} x^{ip+j} \\ &= \sum_{0 \le k \le n} \binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor}\binom{n \bmod p}{k \bmod p} x^k,\end{aligned}$$

or equivalently, by equating coefficients

$$\binom{n}{k} \equiv \binom{\lfloor n/p \rfloor}{\lfloor k/p \rfloor}\binom{n \bmod p}{k \bmod p} \pmod p$$

as we needed to show. □

f) We may prove the equivalence.

Proposition. *If $n = \sum_{0 \le i \le r} a_i p^i$ and $k = \sum_{0 \le i \le r} b_i p^i$, then $\binom{n}{k} \equiv \prod_{0 \le i \le r} \binom{a_i}{b_i}$ (mod p).*

Proof. Let n, k, and p be arbitrary integers such that $n = \sum_{0 \le i \le r} a_i p^i$ and $k = \sum_{0 \le i \le r} b_i p^i$ are the p-ary number representations of n and k with r coefficients a_i, b_i, respectively, $0 \le i \le r$ and $0 \le a_i, b_i \le p$. We must show that

$$\binom{n}{k} \equiv \prod_{0 \le i \le r} \binom{a_i}{b_i} \pmod p.$$

If $r = 0$, then $n = a_0$ and $k = b_0$, and clearly,

$$\begin{aligned} \binom{n}{k} &\equiv \binom{a_0}{b_0} \\ &\equiv \prod_{0 \le i \le r} \binom{a_i}{b_i} \pmod p. \end{aligned}$$

Then, assuming for an arbitrary integer $r \ge 0$ with $n = \sum_{0 \le i \le r} a_i p^i$ and $k = \sum_{0 \le i \le r} b_i p^i$ that

$$\binom{n}{k} \equiv \prod_{0 \le i \le r} \binom{a_i}{b_i} \pmod p,$$

we must show that

$$\binom{n'}{k'} \equiv \prod_{0 \le i \le r+1} \binom{a_i}{b_i} \pmod p$$

for $n' = a_{r+1} p^{r+1} + n = \sum_{0 \le i \le r+1} a_i p^i$ and $k' = b_{r+1} p^{r+1} + k = \sum_{0 \le i \le r+1} b_i p^i$.

But given (e)

$$\begin{aligned}
\binom{n'}{k'} &\equiv \binom{\lfloor n'/p \rfloor}{\lfloor k'/p \rfloor}\binom{n' \bmod p}{k' \bmod p} \\
&\equiv \binom{\left\lfloor \frac{1}{p}\sum_{0\le i\le r+1} a_i p^i \right\rfloor}{\left\lfloor \frac{1}{p}\sum_{0\le i\le r+1} b_i p^i \right\rfloor}\binom{\sum_{0\le i\le r+1} a_i p^i \bmod p}{\sum_{0\le i\le r+1} b_i p^i \bmod p} \\
&\equiv \binom{\left\lfloor \sum_{0\le i\le r+1} a_i p^{i-1} \right\rfloor}{\left\lfloor \sum_{0\le i\le r+1} b_i p^{i-1} \right\rfloor}\binom{a_0}{b_0} \\
&\equiv \binom{\sum_{1\le i\le r+1} a_i p^{i-1}}{\sum_{1\le i\le r+1} b_i p^{i-1}}\binom{a_0}{b_0} \\
&\equiv \binom{\sum_{0\le i\le r} a_{i+1} p^{i}}{\sum_{0\le i\le r} b_{i+1} p^{i}}\binom{a_0}{b_0} \\
&\equiv \binom{a_0}{b_0}\binom{\sum_{0\le i\le r} a_{i+1} p^{i}}{\sum_{0\le i\le r} b_{i+1} p^{i}} \\
&\equiv \binom{a_0}{b_0}\prod_{0\le i\le r}\binom{a_{i+1}}{b_{i+1}} \\
&\equiv \binom{a_0}{b_0}\prod_{1\le i\le r+1}\binom{a_{i}}{b_{i}} \\
&\equiv \prod_{0\le i\le r+1}\binom{a_{i}}{b_{i}} \pmod p
\end{aligned}$$

as we needed to show. □

É. Lucas, *American J. Math.* **1** (1878), 229–230; L. E. Dickson, *Quart. J. Math.* **33** (1902), 383–384; N. J. Fine, *AMM* **54** (1947), 589–592.

▶ **11.** [*M20*] (E. Kummer, 1852.) Let p be prime. Show that if p^n divides

$$\binom{a+b}{a}$$

but p^{n+1} does not, then n is equal to the number of *carries* that occur when a is added to b in the p-ary number system. [*Hint:* See exercise 1.2.5-12.]

Proposition. *If p is prime and $p^n \mid \binom{a+b}{a}$ but $p^{n+1} \nmid \binom{a+b}{a}$, then n is the number of* carries *that occur when a is added to b in the p-ary number system.*

Proof. Let p, n, a, and b be arbitrary nonnegative integers such that p is prime,

$$p^n \mid \binom{a+b}{a},$$

and

$$p^{n+1} \nmid \binom{a+b}{a}.$$

We must show that n is the number of *carries* that occur when a is added to b in the p-ary number system. That is, given representations $a = \sum_{0\le k\le r} a_k p^k$, $b = \sum_{0\le k\le r} b_k p^k$, and $a + b = c = \sum_{0 \le k \le r} c_k p^k$, that n is zero for $a + b < p$ and increases by one for every additional carry.

But

$$\begin{aligned}\binom{a+b}{a} &= \binom{c}{a} \\ &= \frac{c!}{a!b!}\end{aligned}$$

and for μ from exercise 1.2.5-12,

$$\begin{aligned} p^n \mid \frac{c!}{a!b!} \quad &\iff \quad \frac{p^{\mu(c!)}}{p^{\mu(a!)}p^{\mu(b!)}} \mid \frac{c!}{a!b!} \\ &\iff \quad p^{\mu(c!)-\mu(a!)-\mu(b!)} \mid \frac{c!}{a!b!} \\ &\iff \quad n = \mu(c!) - \mu(a!) - \mu(b!). \end{aligned}$$

Then,

$$\begin{aligned} n &= \mu\left(c!\right) - \mu\left(a!\right) - \mu\left(b!\right) \\ &= \frac{c - \sum_{0 \le k \le r} c_k}{p-1} - \frac{a - \sum_{0 \le k \le r} a_k}{p-1} - \frac{b - \sum_{0 \le k \le r} b_k}{p-1} \\ &= \frac{c - \left(\sum_{0 \le k \le r} c_k\right) - a + \left(\sum_{0 \le k \le r} a_k\right) - b + \left(\sum_{0 \le k \le r} b_k\right)}{p-1} \\ &= \frac{-\sum_{0 \le k \le r} c_k + \sum_{0 \le k \le r} a_k + \sum_{0 \le k \le r} b_k}{p-1} \\ &= \frac{\sum_{0 \le k \le r} a_k + b_k - c_k}{p-1}. \end{aligned}$$

To see show that n is the number of *carries*, we construct an inductive argument. As our basis, we consider $a + b < p$, so that $r = 0$, $a = a_0 < p$, $b = b_0 < p$, and $a + b = c = c_0 < p$. In this case, we have no carries, and n is given by

$$n = \frac{\sum_{0 \le k \le r} a_k + b_k - c_k}{p-1} = \frac{a_0 + b_0 - c_0}{p-1} = 0,$$

as expected. Then, assuming n is the number of carries for arbitrary r and $a + b = c$, we must show that $n' = n + 1$ for $a + b = c'$ given a single carry from digit $\kappa - 1$ to κ as a result of the addition of $a_{k-1} + b_{k-1} \ge p$, establishing the relation

$$c'_k = \begin{cases} c_k - p & \text{if } k = \kappa - 1 \\ c_k + 1 & \text{if } k = \kappa \\ c_k & \text{otherwise.} \end{cases}$$

But

$$\begin{aligned}
n' &= \frac{\sum_{0\le k\le r} a_k + b_k - c'_k}{p-1} \\
&= \frac{\sum_{\substack{0\le k\le r \\ \kappa-1\le k\le\kappa}} a_k + b_k - c'_k + \sum_{\substack{0\le k\le r \\ k<\kappa-1\vee\kappa<k}} a_k + b_k - c'_k}{p-1} \\
&= \frac{a_{\kappa-1} + b_{\kappa-1} - c'_{\kappa-1} + a_\kappa + b_\kappa - c'_\kappa + \sum_{\substack{0\le k\le r \\ k<\kappa-1\vee\kappa<k}} a_k + b_k - c'_k}{p-1} \\
&= \frac{a_{\kappa-1} + b_{\kappa-1} - (c_{\kappa-1} - p) + a_\kappa + b_\kappa - (c_\kappa + 1) + \sum_{\substack{0\le k\le r \\ k<\kappa-1\vee\kappa<k}} a_k + b_k - c_k}{p-1} \\
&= \frac{p - 1 + a_{\kappa-1} + b_{\kappa-1} - c_{\kappa-1} + a_\kappa + b_\kappa - c_\kappa + \sum_{\substack{0\le k\le r \\ k<\kappa-1\vee\kappa<k}} a_k + b_k - c_k}{p-1} \\
&= \frac{p - 1 + \sum_{\substack{0\le k\le r \\ \kappa-1\le k\le\kappa}} a_k + b_k - c_k + \sum_{\substack{0\le k\le r \\ k<\kappa-1\vee\kappa<k}} a_k + b_k - c_k}{p-1} \\
&= \frac{p - 1 + \sum_{0\le k\le r} a_k + b_k - c_k}{p-1} \\
&= \frac{\sum_{0\le k\le r} a_k + b_k - c_k}{p-1} + \frac{p-1}{p-1} \\
&= n + 1
\end{aligned}$$

as we needed to show. □

Knuth and Wilf, *Crelle* **396** (1989), 212–219.

12. [*M22*] Are there any positive integers n for which all the nonzero entries in the nth row of Pascal's triangle are *odd*? If so, find all such n.

We want to find all positive integers n such that if $\binom{n}{k} > 0$, then $\binom{n}{k} \equiv 1 \pmod 2$. Let k be an arbitrary integer such that $0 \le k \le n$, and, from Eq. (3), so that $\binom{n}{k} > 0$. We want to find n such that

$$\binom{n}{k} \equiv 1 \pmod 2.$$

But, by exercise 1.2.6-10(f), given the binary representations $n = \sum_{0\le i\le r} a_i 2^i$ and $k = \sum_{0\le i\le r} b_i 2^i$,

$$\begin{aligned}
\binom{n}{k} &\equiv \prod_{0\le i\le r} \binom{a_i}{b_i} \\
&\equiv 1 \pmod 2
\end{aligned}$$

if and only if each $\binom{a_i}{b_i} = 1$. Since for each a_i and b_i, $0 \le a_i, b_i \le 1$, of the four cases, we require $b_i \le a_i$; or equivalently, we require $a_i = 1$ unless $n = k = 0$; or

$$\begin{aligned}
n &= \sum_{0\le i\le r} 2^i \\
&= 2^{r+1} - 1.
\end{aligned}$$

Hence, all the nonzero entries in the nth row of Pascal's triangle—those for which $0 \le k \le n$—are odd if $n = 2^m - 1$ for some integer $m \ge 0$. (This can be generalized to nondivisibility by a prime p if $n = ap^m - 1$ for $1 \le a < p$.)

13. [*M13*] Prove the summation formula, Eq. (10).

Proposition. $\sum_{0\le k\le n}\binom{r+k}{k}=\binom{r+n+1}{n}$.

Proof. Let n and r be arbitrary integers such that $n\ge 0$. We must show that

$$\sum_{0\le k\le n}\binom{r+k}{k}=\binom{r+n+1}{n}.$$

If $n=0$, from Eq. (4),

$$\begin{aligned}\sum_{0\le k\le n}\binom{r+k}{k}&=\sum_{0\le k\le 0}\binom{r+k}{k}\\&=\binom{r+0}{0}\\&=\binom{r}{0}\\&=1\\&=\binom{r+0+1}{0}\\&=\binom{r+n+1}{n}.\end{aligned}$$

Then, assuming

$$\sum_{0\le k\le n}\binom{r+k}{k}=\binom{r+n+1}{n},$$

we must show that

$$\sum_{0\le k\le n+1}\binom{r+k}{k}=\binom{r+n+2}{n+1}.$$

But

$$\begin{aligned}\sum_{0\le k\le n+1}\binom{r+k}{k}&=\binom{r+n+1}{n+1}+\sum_{0\le k\le n}\binom{r+k}{k}\\&=\binom{r+n+1}{n+1}+\binom{r+n+1}{n}\\&=\frac{(r+n+1)!}{(n+1)!r!}+\frac{(r+n+1)!}{n!(r+1)!}\\&=\frac{(r+1)(r+n+1)!}{(n+1)!(r+1)!}+\frac{(n+1)(r+n+1)!}{(n+1)!(r+1)!}\\&=\frac{(r+1)(r+n+1)!+(n+1)(r+n+1)!}{(n+1)!(r+1)!}\\&=\frac{(r+n+2)(r+n+1)!}{(n+1)!(r+1)!}\\&=\frac{(r+n+2)!}{(n+1)!(r+1)!}\\&=\binom{r+n+2}{n+1}\end{aligned}$$

as we needed to show. ∎

14. [*M21*] Evaluate $\sum_{k=0}^{n} k^4$.

For an arbitrary integer $k \geq 0$,

$$\begin{aligned}
k^4 &= \sum_{0\le j\le 4} \left\{ {4 \atop j} \right\} k^{\overline{j}} && \text{from Eq. (45)}\\
&= \sum_{0\le j\le 4} \left\{ {4 \atop j} \right\} j! \binom{k}{j} && \text{from Eq. (3)}\\
&= 24\binom{k}{4} + 36\binom{k}{3} + 14\binom{k}{2} + \binom{k}{1}.
\end{aligned}$$

Summing over k and from Eq. (11),

$$\begin{aligned}
\sum_{0\le k\le n} k^4 &= \sum_{0\le k\le n} \left(24\binom{k}{4} + 36\binom{k}{3} + 14\binom{k}{2} + \binom{k}{1}\right)\\
&= 24\sum_{0\le k\le n}\binom{k}{4} + 36\sum_{0\le k\le n}\binom{k}{3} + 14\sum_{0\le k\le n}\binom{k}{2} + \sum_{0\le k\le n}\binom{k}{1}\\
&= 24\binom{n+1}{5} + 36\binom{n+1}{4} + 14\binom{n+1}{3} + \binom{n+1}{2}\\
&= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}\\
&= \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)\\
&= \frac{n(n+1)(n+\frac{1}{2})(3n^2+3n-1)}{15}.
\end{aligned}$$

15. [*M15*] Prove the binomial formula, Eq. (13).

Proposition. $(x+y)^r = \sum_{0\le k\le r} \binom{r}{k} x^k y^{r-k}$.

Proof. Let x, y, and r be arbitrary integers such that $r \geq 0$. We must show that

$$(x+y)^r = \sum_{0\le k\le r} \binom{r}{k} x^k y^{r-k}.$$

If $r = 0$, then clearly

$$\begin{aligned}
(x+y)^r &= (x+y)^0\\
&= 1\\
&= \binom{0}{0} x^0 y^0\\
&= \sum_{0\le k\le r} \binom{r}{k} x^k y^{r-k}.
\end{aligned}$$

Then, assuming

$$(x+y)^r = \sum_{0\le k\le r} \binom{r}{k} x^k y^{r-k},$$

we must show that

$$(x+y)^{r+1} = \sum_{0\le k\le r+1} \binom{r+1}{k} x^k y^{r+1-k}.$$

But

$$\begin{aligned}
(x+y)^{r+1} &= (x+y)(x+y)^r \\
&= (x+y) \sum_{0\le k\le r} \binom{r}{k} x^k y^{r-k} \\
&= x \sum_{0\le k\le r} \binom{r}{k} x^k y^{r-k} + y \sum_{0\le k\le r} \binom{r}{k} x^k y^{r-k} \\
&= \sum_{0\le k\le r} \binom{r}{k} x^{k+1} y^{r-k} + \sum_{0\le k\le r} \binom{r}{k} x^k y^{r+1-k} \\
&= \sum_{0\le k-1\le r} \binom{r}{k-1} x^k y^{r+1-k} + \sum_{0\le k\le r} \binom{r}{k} x^k y^{r+1-k} \\
&= \sum_{1\le k\le r+1} \binom{r}{k-1} x^k y^{r+1-k} + \sum_{0\le k\le r} \binom{r}{k} x^k y^{r+1-k} \\
&= \sum_{0\le k\le r+1} \binom{r}{k-1} x^k y^{r+1-k} + \sum_{0\le k\le r+1} \binom{r}{k} x^k y^{r+1-k} \\
&= \sum_{0\le k\le r+1} \binom{r+1}{k} x^k y^{r+1-k} && \text{from Eq. (9)}
\end{aligned}$$

as we needed to show. □

16. [*M15*] Given that n and k are positive integers, prove the symmetrical identity

$$(-1)^n \binom{-n}{k-1} = (-1)^k \binom{-k}{n-1}.$$

Proposition. $(-1)^n \binom{-n}{k-1} = (-1)^k \binom{-k}{n-1}$.

Proof. Let n and k be arbitrary positive integers. We must show that

$$(-1)^n \binom{-n}{k-1} = (-1)^k \binom{-k}{n-1}.$$

But

$$\begin{aligned}
(-1)^n \binom{-n}{k-1} &= (-1)^n (-1)^{k-1} \binom{k-1+n-1}{k-1} && \text{from Eq. (17)} \\
&= (-1)^{n+k-1} \binom{n+k-2}{k-1} \\
&= (-1)^k (-1)^{n-1} \binom{n-1+k-1}{k-1} \\
&= (-1)^k (-1)^{n-1} \binom{n-1+k-1}{n-1+k-1-(k-1)} && \text{from Eq. (6)} \\
&= (-1)^k (-1)^{n-1} \binom{n-1+k-1}{n-1} \\
&= (-1)^k \binom{-k}{n-1} && \text{from Eq. (17)}
\end{aligned}$$

as we needed to show. □

▶ **17.** [*M18*] Prove the Chu-Vandermonde formula (21) from Eq. (15), using the idea that $(1+x)^{r+s} = (1+x)^r(1+x)^s$.

Proposition. $\sum_{0\le k\le r}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$.

Proof. Let r and s be arbitrary positive integers. We must show that

$$\sum_{0\le k\le r}\binom{r}{k}\binom{s}{n-k} = \binom{r+s}{n}$$

from Eq. (15)

$$\sum_{0\le k\le r}\binom{r}{k}x^k = (1+x)^r$$

and from the identity

$$(1+x)^{r+s} = (1+x)^r(1+x)^s.$$

But

$$\begin{aligned}\sum_{0\le n\le r+s}\binom{r+s}{n}x^n &= (1+x)^{r+s}\\ &= (1+x)^r(1+x)^s\\ &= \sum_{0\le k\le r}\binom{r}{k}x^k\sum_{0\le k\le s}\binom{s}{k}x^k\\ &= \sum_{0\le k\le r}\binom{r}{k}x^k\sum_{0\le n-k\le s}\binom{s}{n-k}x^{n-k}\\ &= \sum_{0\le n\le r+s}\left(\sum_{0\le k\le r}\binom{r}{k}\binom{s}{n-k}\right)x^n.\end{aligned}$$

Equating coefficients yields

$$\binom{r+s}{n} = \sum_{0\le k\le r}\binom{r}{k}\binom{s}{n-k}$$

as we needed to show. □

18. [*M15*] Prove Eq. (22) using Eqs. (21) and (6).

Proposition. $\sum_k\binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n}$.

Proof. Let m, n, r, and s be arbitrary integers such that $r\ge 0$. We must show that

$$\sum_k\binom{r}{m+k}\binom{s}{n+k} = \binom{r+s}{r-m+n}.$$

But

$$\begin{aligned}
\sum_k \binom{r}{m+k}\binom{s}{n+k} &= \sum_{k-m} \binom{r}{m+k-m}\binom{s}{n+k-m} \\
&= \sum_k \binom{r}{k}\binom{s}{n+k-m} \\
&= \sum_k \binom{r}{k}\binom{s}{k+n-m} \\
&= \sum_k \binom{r}{k}\binom{s}{s-k+m-n} && \text{from Eq. (6)} \\
&= \binom{r+s}{r+s-r+m-n} && \text{from Eq. (21)} \\
&= \binom{r+s}{r-m+n} && \text{from Eq. (6)}
\end{aligned}$$

as we needed to show. □

19. [*M18*] Prove Eq. (23) by induction.

Proposition. $\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} = \binom{s}{n-r}$ *for integers n, $r \geq 0$.*

Proof. Let n, r, and s be arbitrary integers such that $r \geq 0$. We must show that

$$\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} = \binom{s}{n-r}.$$

If $r = 0$

$$\begin{aligned}
\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} &= \sum_k \binom{0}{k}\binom{s+k}{n}(-1)^{-k} \\
&= \binom{0}{0}\binom{s+0}{n}(-1)^{0} \\
&= \binom{s}{n} \\
&= \binom{s}{n-r}.
\end{aligned}$$

Then, assuming

$$\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} = \binom{s}{n-r},$$

we must show that

$$\sum_k \binom{r+1}{k}\binom{s+k}{n}(-1)^{r+1-k} = \binom{s}{n-r-1}.$$

But

$$\begin{aligned}
&\sum_k \binom{r+1}{k}\binom{s+k}{n}(-1)^{r+1-k} \\
&\quad= \sum_k \left(\binom{r}{k} + \binom{r}{k-1}\right)\binom{s+k}{n}(-1)^{r+1-k} \\
&\quad= \sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r+1-k} + \sum_k \binom{r}{k-1}\binom{s+k}{n}(-1)^{r+1-k} \\
&\quad= -\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} + \sum_{k+1} \binom{r}{k}\binom{s+k+1}{n}(-1)^{r-k} \\
&\quad= -\sum_k \binom{r}{k}\binom{s+k}{n}(-1)^{r-k} + \sum_{k} \binom{r}{k}\binom{s+k+1}{n}(-1)^{r-k} \\
&\quad= -\binom{s}{n-r} + \binom{s+1}{n-r} \\
&\quad= \binom{s+1}{n-r} - \binom{s}{n-r} \\
&\quad= \frac{s+1}{s+1-(n-r)}\binom{s}{n-r} - \binom{s}{n-r} \\
&\quad= \left(\frac{s+1}{s+1-(n-r)} - 1\right)\binom{s}{n-r} \\
&\quad= \left(\frac{n-r}{s+1-n+r}\right)\binom{s}{n-r} \\
&\quad= \frac{n-r}{s-n+r+1}\,\frac{s!}{(n-r)!(s-n+r)!} \\
&\quad= \frac{s!}{(n-r-1)!(s-n+r+1)!} \\
&\quad= \binom{s}{n-r-1}
\end{aligned}$$

as we needed to show. □

20. [*M20*] Prove Eq. (24) by using Eqs. (21) and (19), then show that another use of Eq. (19) yields Eq. (25).

We may prove Eq. (24) using Eqs. (19) and (21).

Proposition. $\sum_{0\le k\le r}\binom{r-k}{m}\binom{s}{k-t}(-1)^{k-t} = \binom{r-t-s}{r-t-m}$ *for nonnegative integers* t*,* r*, and* m*.*

Proof. Let r, m, s, and t be arbitrary integers such that $r, m, t \ge 0$. We must show that

$$\sum_{0\le k\le r}\binom{r-k}{m}\binom{s}{k-t}(-1)^{k-t} = \binom{r-t-s}{r-t-m}.$$

But

$$\begin{aligned}
\sum_{0\le k\le r}\binom{r-k}{m}\binom{s}{k-t}(-1)^{k-t} &= \sum_{0\le k\le r}(-1)^{r-k-m}\binom{-(m+1)}{r-k-m}\binom{s}{k-t}(-1)^{k-t} && \text{from Eq. (19)}\\
&= \sum_{0\le k\le r}(-1)^{r-m-t}\binom{-(m+1)}{r-k-m}\binom{s}{k-t}\\
&= (-1)^{r-m-t}\sum_{0\le k\le r}\binom{-(m+1)}{r-k-m}\binom{s}{k-t}\\
&= (-1)^{r-m-t}\sum_{-t\le k-t\le r-t}\binom{-(m+1)}{r-k-t-m}\binom{s}{k}\\
&= (-1)^{r-t-m}\sum_{k}\binom{s}{k}\binom{-(m+1)}{r-t-m-k}\\
&= (-1)^{r-t-m}\binom{s-m-1}{r-t-m} && \text{from Eq. (21)}\\
&= (-1)^{r-t-m}\binom{r-t-m-r+t+s-1}{r-t-m}\\
&= \binom{r-t-s}{r-t-m} && \text{from Eq. (17)}
\end{aligned}$$

as we needed to show. □

We may also show that another use of Eq. (19) yields Eq. (25).

Proposition. $\sum_{0\le k\le r}\binom{r-k}{m}\binom{s+k}{n}=\binom{r+s+1}{m+n+1}$ *for nonnegative integers* m*,* n*,* r*, and* s*.*

Proof. Let m, n, r, and s be arbtirary nonnegative integers. We must show that

$$\sum_{0\le k\le r}\binom{r-k}{m}\binom{s+k}{n}=\binom{r+s+1}{m+n+1}.$$

But

$$\begin{aligned}
\sum_{0\le k\le r}\binom{r-k}{m}\binom{s+k}{n} &= \sum_{0\le k\le r}\binom{r-k}{m}(-1)^{s+k-n}\binom{-(n+1)}{s+k-n} && \text{from Eq. (19)}\\
&= \sum_{0\le k\le r}\binom{r-k}{m}\binom{-(n+1)}{k-(n-s)}(-1)^{k-(n-s)}\\
&= \binom{r-(n-s)+n+1}{r-(n-s)-m} && \text{from Eq. (24)}\\
&= \binom{r+s+1}{r+s+1-m-n-1}\\
&= \binom{r+s+1}{m+n+1} && \text{from Eq. (6)}
\end{aligned}$$

as we needed to show. □

▶ **21.** [*M05*] Both sides of Eq. (25) are polynomials in s; why isn't that equation an identity in s?

According to the text on page 57, any polynomial $\sum_{0\le k\le d}a_k s^k$ can be expressed as $\sum_{0\le k\le d}b_k\binom{s}{k}$ for suitably chosen coefficients $b_0, b_1, \cdots, b_d$.

And so, the left hand of Eq. (25) can be expressed as

$$\sum_{0\le k\le r} \binom{r-k}{m}\binom{s+k}{n} = \sum_{0\le k\le n} a_k s^k;$$

that is, a polynomial of degree at most n for suitably chosen coefficients $a_0, a_1, \cdots, a_n$; and the right hand of Eq. (25) can be expressed as

$$\binom{r+s+1}{m+n+1} = \sum_{0\le k\le m+n+1} a'_k s^k;$$

that is, a polynomial of degree at most $m+n+1$ for suitably chosen coefficients $a'_0, a'_1, \cdots, a'_{m+n+1}$.

Therefore, even though both sides of Eq. (25) are polynomials in s, since they do not agree at all $m+n+1$ possible points, the equation does not serve as an identity in s.

22. [*M20*] Prove Eq. (26) for the special case $s = n-1-r+nt$.

Proposition. $\sum_{0\le k} \binom{r-tk}{k}\binom{n-1-(r-tk)}{n-k}\frac{r}{r-tk} = \binom{n-1}{n}$ *for integers* n.

Proof. Let r, t, and n be arbtirary integers. We must show that

$$\sum_{0\le k} \binom{r-tk}{k}\binom{n-1-(r-tk)}{n-k}\frac{r}{r-tk} = \binom{n-1}{n}.$$

We consider two cases, depending on whether $k \le r-tk$ or not.

Case 1. [$k \le r-tk$] In the case that $k \le r-tk$,

$$\begin{aligned} k \le r-tk &\implies -(r-tk) \le -k \\ &\implies n-(r-tk) \le n-k \\ &\implies n-1-(r-tk) < n-k \\ &\implies \binom{n-1-(r-tk)}{n-k} = 0 \end{aligned}$$

which gives us that

$$\sum_{0\le k} \binom{r-tk}{k}\binom{n-1-(r-tk)}{n-k}\frac{r}{r-tk} = 0 = \binom{n-1}{n}$$

in this case.

Case 2. [$k > r-tk$] In the case that $k > r-tk$, clearly

$$r-tk < k \implies \binom{r-tk}{k} = 0$$

which gives us that

$$\sum_{0\le k} \binom{r-tk}{k}\binom{n-1-(r-tk)}{n-k}\frac{r}{r-tk} = 0 = \binom{n-1}{n}$$

in this case.

Therefore, in either case, we have that

$$\sum_{0\le k} \binom{r-tk}{k}\binom{n-1-(r-tk)}{n-k}\frac{r}{r-tk} = 0 = \binom{n-1}{n}$$

as we needed to show. □

23. [*M13*] Assuming that Eq. (26) holds for (r, s, t, n) and $(r, s-t, t, n-1)$, prove it for $(r, s+1, t, n)$.

Proposition. *If* $\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} = \binom{r-s-tn}{n}$ *and* $\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k-1}\frac{r}{r-tk} = \binom{r-s-tn}{n-1}$, *then* $\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)+1}{n-k}\frac{r}{r-tk} = \binom{r-s-tn+1}{n}$.

Proof. Let r, s, t, and n be arbitrary integers such that

$$\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} = \binom{r-s-tn}{n}$$

and

$$\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k-1}\frac{r}{r-tk} = \binom{r-s-tn}{n-1}.$$

We must show that

$$\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)+1}{n-k}\frac{r}{r-tk} = \binom{r-s-tn+1}{n}.$$

But

$$\begin{aligned}
&\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)+1}{n-k}\frac{r}{r-tk}\\
&\quad= \sum_{0\le k}\binom{r-tk}{k}\left(\binom{s-t(n-k)}{n-k}+\binom{s-t(n-k)}{n-k-1}\right)\frac{r}{r-tk}\\
&\quad= \sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} + \sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k-1}\frac{r}{r-tk}\\
&\quad= \binom{r-s-tn}{n}+\binom{r-s-tn}{n-1}\\
&\quad= \binom{r-s-tn+1}{n}
\end{aligned}$$

as we needed to show. □

24. [*M15*] Explain why the results of the previous two exercises combine to give a proof of Eq. (26).

The results of the previous two exercises combine to give a proof of Eq. (26) as a proof by induction on n.

In the case that $n < 0$,

$$\begin{aligned}
\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} &= 0\\
&= \binom{r+s-tn}{n};
\end{aligned}$$

and in the case that $n = 0$,

$$\begin{aligned}\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} &= \binom{r}{0}\binom{s}{0}\frac{r}{r}\\ &= 1\\ &= \binom{r+s}{0}\\ &= \binom{r+s-tn}{n}.\end{aligned}$$

Otherwise, in the case that $n > 0$, we may construct a proof by induction on $m \ge -1$ with $s = n - r + nt + m$. If $m = -1$,

$$\begin{aligned}&\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk}\\ &\quad= \sum_{0\le k}\binom{r-tk}{k}\binom{n-r+nt+m-t(n-k)}{n-k}\frac{r}{r-tk}\\ &\quad= \sum_{0\le k}\binom{r-tk}{k}\binom{n-1-r+nt-t(n-k)}{n-k}\frac{r}{r-tk}\\ &\quad= \sum_{0\le k}\binom{r-tk}{k}\binom{n-1-(r-tk)}{n-k}\frac{r}{r-tk}\\ &\quad= \binom{n-1}{n} && \text{by exercise 22}\\ &\quad= \binom{n+m}{n}\\ &\quad= \binom{r+n-r+nt+m-tn}{n}\\ &\quad= \binom{r+s-tn}{n}.\end{aligned}$$

Then, assuming

$$\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} = \binom{r-s-tn}{n}$$

and

$$\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k-1}\frac{r}{r-tk} = \binom{r-s-tn}{n-1},$$

we must show that

$$\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)+1}{n-k}\frac{r}{r-tk} = \binom{r-s-tn+1}{n}.$$

But, by exercise 24

$$\sum_{0\le k}\binom{r-tk}{k}\binom{s-t(n-k)+1}{n-k}\frac{r}{r-tk} = \binom{r-s-tn+1}{n}$$

and hence the result.

25. [*HM30*] Let the polynomial $A_n(x,t)$ be defined as in Eq. (30). Let $z = x^{t+1} - x^t$. Prove that $\sum_k A_k(r,t)z^k = x^r$, provided z is small enough. [*Note:* If $t = 0$, this result is essentially the binomial theorm, and this equation is an important generalization of the binomial theorem. The binomial theorem (15) may be assumed in the proof.] *Hint:* Start with the identity

$$\sum_j (-1)^j \binom{k}{j}\binom{r-jt}{k}\frac{r}{r-jt} = \delta_{k0}.$$

Proposition. $\sum_k A_k(r,t)z^k = x^r$ *for* $z = x^{t+1} - x^t$ *sufficiently small.*

Proof. Let $A_n(x,t)$ be the nth degree polynomial in x that satisfies

$$A_n(x,t) = \binom{x-nt}{n}\frac{x}{x-nt}$$

for $x \neq nt$; and $z = x^{t+1} - x^t$ sufficiently small. We must show that

$$\sum_j A_j(r,t)z^j = x^r.$$

We may first prove that the sum converges by using the ratio test; that is, that

$$\lim_{k\to\infty}\left|\frac{A_{k+1}(r,t)z^{k+1}}{A_k(r,t)z^k}\right| < 1.$$

But if z is sufficiently small, $|z| \leq 1/t$, or equivalently, $|-tz| < 1$, then

$$\begin{aligned}
1 &> |-tz| \\
&= \lim_{k\to\infty}\left|\frac{(r-kt)}{(k+1)}z\right| \\
&= \lim_{k\to\infty}\left|\prod_{1\leq j\leq k}\frac{(r-kt)(r-(k+1)t)(r-kt+1-j)}{(r-(k+1)t)(k+1)(r-kt+1-j)}z\right| \\
&\geq \lim_{k\to\infty}\left|\prod_{1\leq j\leq k}\frac{(r-kt)(r-(k+1)t-k)(r-(k+1)t+1-j)}{(r-(k+1)t)(k+1)(r-kt+1-j)}z\right| \\
&= \lim_{k\to\infty}\left|\frac{(r-kt)z(r-(k+1)t-k)\prod_{1\leq j\leq k}(r-(k+1)t+1-j)}{(r-(k+1)t)(k+1)\prod_{1\leq j\leq k}(r-kt+1-j)}\right| \\
&= \lim_{k\to\infty}\left|\frac{(r-kt)z\frac{r-(k+1)t-k}{k+1}\prod_{1\leq j\leq k}\frac{r-(k+1)t+1-j}{j}}{(r-(k+1)t)\prod_{1\leq j\leq k}\frac{r-kt+1-j}{j}}\right| \\
&= \lim_{k\to\infty}\left|\frac{(r-kt)z\prod_{1\leq j\leq k+1}\frac{r-(k+1)t+1-j}{j}}{(r-(k+1)t)\prod_{1\leq j\leq k}\frac{r-kt+1-j}{j}}\right| \\
&= \lim_{k\to\infty}\left|\frac{\binom{r-(k+1)t}{k+1}(r-kt)z}{\binom{r-kt}{k}(r-(k+1)t)}\right| \\
&= \lim_{k\to\infty}\left|\frac{\binom{r-(k+1)t}{k+1}\frac{r}{r-(k+1)t}z}{\binom{r-kt}{k}\frac{r}{r-kt}}\right| \\
&= \lim_{k\to\infty}\left|\frac{A_{k+1}(r,t)z}{A_k(r,t)}\right| \\
&= \lim_{k\to\infty}\left|\frac{A_{k+1}(r,t)z^{k+1}}{A_k(r,t)z^k}\right|
\end{aligned}$$

and hence the proof of convergence.

Then, given the identity

$$\sum_j (-1)^j \binom{k}{j} \binom{r-jt}{k} \frac{r}{r-jt} = \delta_{k0},$$

and letting $x = 1/(1+w)$ so that $z = -w/(1+w)^{t+1} = x^{t+1} - x^t$, we have that

$$\begin{aligned}
1 &= \delta_{00} \\
&= \sum_k \delta_{k0} w^k \\
&= \sum_k \sum_j (-1)^j \binom{k}{j} \binom{r-jt}{k} \frac{r}{r-jt} w^k \\
&= \sum_j (-1)^j \frac{r}{r-jt} \sum_k \binom{k}{j} \binom{r-jt}{k} w^k \\
&= \sum_j (-1)^j \frac{r}{r-jt} \sum_k \binom{r-jt}{j} \binom{r-jt-j}{k-j} w^k && \text{from Eq. (2)} \\
&= \sum_j (-1)^j \binom{r-jt}{j} \frac{r}{r-jt} \sum_k \binom{r-jt-j}{k-j} w^k \\
&= \sum_j (-1)^j A_j(r,t) \sum_k \binom{r-jt-j}{k-j} w^k \\
&= \sum_j (-1)^j A_j(r,t) \sum_k \binom{r-jt-j}{k-j} w^{k-j} w^j \\
&= \sum_j (-1)^j A_j(r,t) (1+w)^{r-jt-j} w^j \\
&= \sum_j (-1)^j A_j(r,t) (1/x)^{r-jt-j} (1/x - 1)^j \\
&= \sum_j A_j(r,t) (-1)^j (1/x)^{r-jt-j} (1/x - 1)^j \\
&= \sum_j A_j(r,t) (-1)^j (1/x - 1)^j (1/x)^{r-jt-j} \\
&= \sum_j A_j(r,t) (-1)^j (1/x - 1)^j (1/x)^{-jt-j} (1/x)^r \\
&= \sum_j A_j(r,t) (-(1/x - 1))^j ((1/x)^{-t-1})^j (1/x)^r \\
&= \sum_j A_j(r,t) ((1 - 1/x) x^{t+1})^j (1/x)^r \\
&= \sum_j A_j(r,t) (x^{t+1} - x^t)^j (1/x)^r \\
&= \sum_j A_j(r,t) z^j (1/x)^r;
\end{aligned}$$

or equivalently, that

$$\sum_j A_j(r,t) z^j = x^r$$

as we needed to show. □

H. W. Gould, *AMM* **63** (1956), 84–91.

26. [*HM25*] Using the assumptions of the previous exercise, prove that

$$\sum_k \binom{r-tk}{k} z^k = \frac{x^{r+1}}{(t+1)x-t}.$$

Proposition. $\sum_k \binom{r-tk}{k} z^k = \frac{x^{r+1}}{(t+1)x-t}$.

Proof. Let $A_n(x,t)$ be the nth degree polynomial in x from exercise 25 that satisfies

$$A_n(x,t) = \binom{x-nt}{n} \frac{x}{x-nt}$$

for $x \neq nt$; and $z = x^{t+1} - x^t$ sufficiently small so that $|z| \leq 1/t$. We must show that

$$\sum_k \binom{r-tk}{k} z^k = \frac{x^{r+1}}{(t+1)x-t}.$$

From exercise 25, we have that $\sum_k A_k(r,t) z^k = x^r$, or equivalently that

$$\begin{aligned}
1 &= \sum_k A_k(r,t) z^k x^{-r} \\
&= \sum_k A_k(r,t) (x^{t+1} - x^t)^k x^{-r} \\
&= \sum_k A_k(r,t) x^{tk-r} (x-1)^k;
\end{aligned}$$

we have by definition that

$$\begin{aligned}
1 &= \frac{dz}{dz} \\
&= \frac{d}{dz}(x^{t+1} - x^t) \\
&= ((t+1)x^t - tx^{t-1}) \frac{dx}{dz} \\
&= x^{t-1}((t+1)x - t) \frac{dx}{dz}
\end{aligned}$$

or equivalently that

$$\frac{dx}{dz} = \frac{x}{x^t((t+1)x-t)};$$

and we also have that $\frac{d}{dz} \sum_k A_k(r,t) z^k = \sum_k k A_k(r,t) z^{k-1} = \frac{d(x^r)}{dz}$, or equivalently that

$$\begin{aligned}\sum_k kA_k(r,t)z^k &= z\frac{d(x^r)}{dz}\\ &= (x^{t+1}-x^t)rx^{r-1}\frac{dx}{dz}\\ &= (x^{t+1}-x^t)rx^{r-1}\frac{x}{x^t((t+1)x-t)}\\ &= rx^{r-1}(\frac{x^2}{(t+1)x-t}-\frac{x}{(t+1)x-t})\\ &= rx^r\frac{x-1}{(t+1)x-t}.\end{aligned}$$

Finally, diffirentiating the first equality yields

$$\begin{aligned}&\frac{d}{dx}1 = 0\\ &= \frac{d}{dx}\left(\sum_k A_k(r,t)x^{tk-r}(x-1)^k\right)\\ &= \sum_k A_k(r,t)\left((x-1)^k(tk-r)(x^{tk-r-1})+x^{tk-r}k(x-1)^{k-1}\right)\\ &= \sum_k A_k(r,t)\left((tk-r)(x^{-r-1})+x^{-r}k(x-1)^{-1}\right)(x-1)^kx^{tk}\\ &= \sum_k A_k(r,t)\left((tk-r)(x^{-r-1})+x^{-r}k(x-1)^{-1}\right)z^k\\ &= \sum_k(tk-r)(x^{-r-1})A_k(r,t)z^k+\sum_k x^{-r}k(x-1)^{-1}A_k(r,t)z^k\\ &= \sum_k(tk-r)(x^{-r-1})\binom{r-kt}{k}\frac{r}{r-kt}z^k+\sum_k x^{-r}k(x-1)^{-1}A_k(r,t)z^k\\ &= \sum_k(-r)(x^{-r-1})\binom{r-kt}{k}z^k+\sum_k x^{-r}k(x-1)^{-1}A_k(r,t)z^k\\ &= -rx^{-1}\sum_k\binom{r-kt}{k}z^k+(x-1)^{-1}\sum_k kA_k(r,t)z^k\\ &= -rx^{-1}\sum_k\binom{r-kt}{k}z^k+(x-1)^{-1}rx^r\frac{x-1}{(t+1)x-t}\\ &= -rx^{-1}\sum_k\binom{r-kt}{k}z^k+\frac{rx^r}{(t+1)x-t}\end{aligned}$$

if and only if

$$\begin{aligned}\sum_k\binom{r-kt}{k}z^k &= \frac{x}{r}\frac{rx^r}{(t+1)x-t}\\ &= \frac{x^{r+1}}{(t+1)x-t}\end{aligned}$$

as we needed to show. □

27. [*HM21*] Solve Example 4 in the text by using the result of exercise 25, and prove Eq. (26) from the preceding two exercises. [*Hint:* See exercise 17.]

We may solve Example 4 from the text using the result of exercise 25.

Proposition. $\sum_k A_k(r,t)A_{n-k}(s,t) = A_n(r+s,t)$ *for nonnegative integers* n.

Proof. Let $A_n(x,t)$ be the nth degree polynomial in x from exercise 25 that satisfies

$$A_n(x,t) = \binom{x-nt}{n}\frac{x}{x-nt}$$

for $x \neq nt$; and $z = x^{t+1} - x^t$ sufficiently small so that $|z| \leq 1/t$. We must show that

$$\sum_k A_k(r,t)A_{n-k}(s,t) = A_n(r+s,t).$$

From exercise 25, we have that $\sum_n A_n(r+s,t)z^n = x^{r+s}$. And so,

$$\begin{aligned}
\sum_n \sum_k A_k(r,t)A_{n-k}(s,t)z^n &= \sum_n \sum_k \sum_{\substack{j \\ j=n-k}} A_k(r,t)A_j(s,t)z^n \\
&= \sum_k \sum_{\substack{j \\ j=n-k}} A_k(r,t)A_j(s,t)z^{j+k} \\
&= \sum_k A_k(r,t)z^k \sum_j A_j(s,t)z^j \\
&= x^r x^s \\
&= x^{r+s} \\
&= \sum_n A_n(r+s,t)z^n = x^{r+s}.
\end{aligned}$$

Equating coefficients yields

$$\sum_k A_k(r,t)A_{n-k}(s,t) = A_n(r+s,t)$$

as we needed to show. ☐

We may also prove Eq. (26) from the preceding two exercises.

Proposition. $\sum_{k\geq 0}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} = \binom{r+s-tn}{n}$.

Proof. Let n be an arbitrary integer. We must show that

$$\sum_{k\geq 0}\binom{r-tk}{k}\binom{s-t(n-k)}{n-k}\frac{r}{r-tk} = \binom{r+s-tn}{n}.$$

From exercise 25, we have that $\sum_{k\geq 0} A_k(r,t)z^k = x^r$ and from exercise 26, we have that $\sum_j \binom{s-tj}{j} z^j = \frac{x^{s+1}}{(t+1)x-t}$. Multiplying both equations yields

$$\begin{aligned}
x^r \frac{x^{s+1}}{(t+1)x-t} &= \frac{x^{r+s+1}}{(t+1)x-t} \\
&= \sum_{k\geq 0} A_k(r,t) z^k \sum_j \binom{s-tj}{j} z^j \\
&= \sum_{k\geq 0} A_k(r,t) z^k \sum_{\substack{n \\ j=n-k}} \binom{s-t(n-k)}{n-k} z^{n-k} \\
&= \sum_n \sum_{k\geq 0} A_k(r,t) \binom{s-t(n-k)}{n-k} z^n \\
&= \sum_n \sum_{k\geq 0} \binom{r-kt}{k} \frac{r}{r-kt} \binom{s-t(n-k)}{n-k} z^n; \\
&= \sum_n \sum_{k\geq 0} \binom{r-kt}{k} \binom{s-t(n-k)}{n-k} \frac{r}{r-kt} z^n;
\end{aligned}$$

and again, from exercise 26,

$$\frac{x^{r+s+1}}{(t+1)x-t} = \sum_n \binom{r+s-tn}{n} z^n,$$

which gives us the equality

$$\sum_n \sum_{k\geq 0} \binom{r-kt}{k} \binom{s-t(n-k)}{n-k} \frac{r}{r-kt} z^n = \sum_n \binom{r+s-tn}{n} z^n.$$

Finally, equating coefficients yields

$$\sum_{k\geq 0} \binom{r-kt}{k} \binom{s-t(n-k)}{n-k} \frac{r}{r-kt} = \binom{r+s-tn}{n},$$

and hence the result. □

28. [*M25*] Prove that

$$\sum_k \binom{r+tk}{k} \binom{s-tk}{n-k} = \sum_{k\geq 0} \binom{r+s-k}{n-k} t^k,$$

if n is a nonnegative integer.

Proposition. $\sum_k \binom{r+tk}{k}\binom{s-tk}{n-k} = \sum_{k\geq 0} \binom{r+s-k}{n-k} t^k$ *for n a nonnegative integer.*

Proof. Let n be an arbitrary nonnegative integer. We must show that

$$\sum_k \binom{r+tk}{k} \binom{s-tk}{n-k} = \sum_{k\geq 0} \binom{r+s-k}{n-k} t^k.$$

If $n = 0$,

$$\begin{aligned}\sum_k \binom{r+tk}{k}\binom{s-tk}{n-k} &= \sum_k \binom{r+tk}{k}\binom{s-tk}{-k}\\ &= 0\\ &= \sum_{k\geq 0}\binom{r+s-k}{-k}\\ &= \sum_{k\geq 0}\binom{r+s-k}{-k}t^k.\end{aligned}$$

Then, assuming

$$\sum_k \binom{r+tk}{k}\binom{s-tk}{n-k} = \sum_{k\geq 0}\binom{r+s-k}{n-k}t^k$$

we must show that

$$\sum_k \binom{r+tk}{k}\binom{s-tk}{n+1-k} = \sum_{k\geq 0}\binom{r+s-k}{n+1-k}t^k.$$

But from Eq. (26),

$$\begin{aligned}
\sum_k \binom{r+tk}{k}\binom{s-tk}{n+1-k} &= \sum_k \binom{r+tk}{k}\binom{s-tk}{n+1-k}\frac{r+tk}{r+tk} \\
&= \sum_k \binom{r+tk}{k}\binom{s-tk}{n+1-k}\frac{r}{r+tk} \\
&\quad + \sum_k \binom{r+tk}{k}\binom{s-tk}{n+1-k}\frac{tk}{r+tk} \\
&= \sum_k \binom{r-(-t)k}{k}\binom{(s-tn)-(-t)(n-k)}{n+1-k}\frac{r}{r-(-t)k} \\
&\quad + \sum_k \binom{r+tk}{k}\binom{s-tk}{n+1-k}\frac{tk}{r+tk} \\
&= \binom{r+(s-tn)-(-t)n}{n+1} \\
&\quad + \sum_k \binom{r+tk}{k}\binom{s-tk}{n+1-k}\frac{tk}{r+tk} \\
&= \binom{r+s}{n+1} + t\sum_k \frac{k}{r+tk}\binom{r+tk}{k}\binom{s-tk}{n+1-k} \\
&= \binom{r+s}{n+1} + t\sum_k \binom{r+tk-1}{k-1}\binom{s-tk}{n+1-k} \\
&= \binom{r+s}{n+1} + t\sum_k \binom{r+t(k+1)-1}{k+1-1}\binom{s-t(k+1)}{n+1-(k+1)} \\
&= \binom{r+s}{n+1} + t\sum_k \binom{r+t-1+tk}{k}\binom{s-t-tk}{n-k} \\
&= \binom{r+s}{n+1} + t\sum_{k\ge 0} \binom{r+t-1+s-t-k}{n-k}t^k \\
&= \binom{r+s}{n+1} + t\sum_{k\ge 0} \binom{r+s-k-1}{n-k}t^k \\
&= \binom{r+s}{n+1} + \sum_{k\ge 0} \binom{r+s-(k+1)}{n+1-(k+1)}t^{k+1} \\
&= \binom{r+s-0}{n+1-0}t^0 + \sum_{k\ge 1} \binom{r+s-k}{n+1-k}t^k \\
&= \sum_{k\ge 0} \binom{r+s-k}{n+1-k}t^k
\end{aligned}$$

as we needed to show. □

29. [*M20*] Show that Eq. (34) is just a special case of the general identity proved in exercise 1.2.3-33.

Eq. (34) for $r \ge 0$ is

$$\sum_k \binom{r}{k}(-1)^{r-k}\sum_{0\le j\le r} b_j k^j = r!b_r,$$

while the general identity proved in exercise 1.2.3-33 is

$$\sum_{1\le k\le r} \frac{k^j}{\prod_{\substack{1\le i\le r\\ i\ne k}}(k-i)} = \begin{cases} 0 & \text{if } 0\le j<r-1\\ 1 & \text{if } j=r-1\\ \sum_{1\le k\le r} k & \text{if } j=r.\end{cases}$$

Thus,

$$\begin{aligned}
\sum_k \binom{r}{k}(-1)^{r-k}\sum_{0\le j\le r} b_j k^j &= \sum_{0\le k\le r}\binom{r}{k}(-1)^{r-k}\sum_{0\le j\le r} b_j k^j\\
&= \sum_{0\le j\le r} b_j \sum_{0\le k\le r}\binom{r}{k}(-1)^{r-k}k^j\\
&= \sum_{0\le j\le r} b_j \sum_{0\le k\le r}\frac{r!}{k!(r-k)!}(-1)^{r-k}k^j\\
&= r!\sum_{0\le j\le r} b_j \sum_{0\le k\le r}\frac{k^j}{(-1)^{r-k}k!(r-k)!}\\
&= r!\sum_{0\le j\le r} b_j \sum_{0\le k\le r}\frac{k^j}{\prod_{1\le i\le r-k}(-1)\prod_{1\le i\le k} i\prod_{1\le i\le r-k} i}\\
&= r!\sum_{0\le j\le r} b_j \sum_{0\le k\le r}\frac{k^j}{\prod_{1\le i\le k} i\prod_{1\le i\le r-k}(-i)}\\
&= r!\sum_{0\le j\le r} b_j \sum_{0\le k\le r}\frac{k^j}{\prod_{0\le i\le k-1}(k-i)\prod_{k+1\le i\le r}(k-i)}\\
&= r!\sum_{0\le j\le r} b_j \sum_{0\le k\le r}\frac{k^j}{\prod_{\substack{0\le i\le r\\ i\ne k}}(k-i)}\\
&= r!\sum_{-1\le j\le r-1} b_{j+1} \sum_{1\le k\le r}\frac{k^j}{\prod_{\substack{1\le i\le r\\ i\ne k}}(k-i)}\\
&= r!b_{r-1+1}(1)\\
&= r!b_r
\end{aligned}$$

and hence the result.

▶ **30.** [*M24*] Show that there is a better way to solve Example 3 than the way used in the text, by manipulating the sum so that Eq. (26) applies.

We wish to evaluate the sum from Example 3

$$\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}$$

for positive integers m and n, using Eq. (26)

$$\sum_{k\ge 0}\binom{1+2k}{k}\binom{-m-1-2k}{n-m-k}\frac{1}{1+2k} = \binom{-m}{n-m}.$$

But

$$
\begin{aligned}
&\sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \\
&\quad= \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1}\frac{2k+1}{2k+1} \\
&\quad= \sum_k \binom{n+k}{m+2k}\frac{2k+1}{k+1}\binom{2k}{k}\frac{(-1)^k}{2k+1} \\
&\quad= \sum_{k\ge 0} \binom{n+k}{m+2k}\binom{2k+1}{k+1}\frac{(-1)^k}{2k+1} && \text{from Eq. (7)} \\
&\quad= \sum_{k\ge 0} \binom{n+k}{m+2k}\binom{2k+1}{k}\frac{(-1)^k}{2k+1} && \text{from Eq. (6)} \\
&\quad= \sum_{k\ge 0} (-1)^{n+k-m-2k}\binom{-(m+2k+1)}{n+k-m-2k}\binom{2k+1}{k}\frac{(-1)^k}{2k+1} && \text{from Eq. (19)} \\
&\quad= \sum_{k\ge 0} \binom{-m-2k-1}{n-m-k}\binom{2k+1}{k}\frac{(-1)^{n-m-k}(-1)^k}{2k+1} \\
&\quad= \sum_{k\ge 0} \binom{-m-2k-1}{n-m-k}\binom{2k+1}{k}\frac{(-1)^{n-m}}{2k+1} \\
&\quad= \sum_{k\ge 0} \binom{-m-2k-1}{n-m-k}\binom{2k+1}{k}\frac{(-1)^{n-m}}{2k+1} \\
&\quad= \sum_{k\ge 0} \binom{1+2k}{k}\binom{-m-1-2k}{n-m-k}\frac{(-1)^{n-m}}{1+2k} \\
&\quad= (-1)^{n-m}\binom{-m}{n-m} && \text{from Eq. (26)} \\
&\quad= (-1)^{n-m}\binom{n-m-n+1-1}{n-m} \\
&\quad= \binom{n-1}{n-m} && \text{from Eq. (17)} \\
&\quad= \binom{n-1}{n-1-n+m} && \text{from Eq. (6)} \\
&\quad= \binom{n-1}{m-1}
\end{aligned}
$$

and hence the result.

▶ **31.** [*M20*] Evaluate

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n}$$

in terms of r, s, m, and n, given that m and n are integers. Begin by replacing

$$\binom{r+k}{m+n} \quad \text{by} \quad \sum_j \binom{r}{m+n-j}\binom{k}{j}.$$

We have

$$
\begin{aligned}
&\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} \\
&\quad= \sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\sum_j \binom{r}{m+n-j}\binom{k}{j} \\
&\quad= \sum_j\sum_k \binom{m-r+s}{k}\binom{k}{j}\binom{n+r-s}{n-k}\binom{r}{m+n-j} \\
&\quad= \sum_j\sum_k \binom{m-r+s}{j}\binom{m-r+s-j}{k-j}\binom{n+r-s}{n-k}\binom{r}{m+n-j} && \text{from Eq. (20)} \\
&\quad= \sum_j\sum_k \binom{m-r+s}{j}\binom{m-r+s-j}{m-r+s-k}\binom{n+r-s}{n-k}\binom{r}{m+n-j} && \text{from Eq. (6)} \\
&\quad= \sum_j\sum_k \binom{m-r+s}{j}\binom{m-r+s-j}{m-r+s-k}\binom{n+r-s}{r-s+k}\binom{r}{m+n-j} && \text{from Eq. (6)} \\
&\quad= \sum_j \binom{m-r+s}{j}\binom{r}{m+n-j}\sum_k \binom{m-r+s-j}{m-(k+r-s)}\binom{n+r-s}{k+r-s} \\
&\quad= \sum_j \binom{m-r+s}{j}\binom{r}{m+n-j}\sum_k \binom{n+r-s}{k}\binom{m-r+s-j}{m-k} \\
&\quad= \sum_j \binom{m-r+s}{j}\binom{r}{m+n-j}\binom{n+m-j}{m} && \text{from Eq. (21)} \\
&\quad= \sum_j \binom{m-r+s}{j}\binom{r}{m}\binom{r-m}{n-j} && \text{from Eq. (20)} \\
&\quad= \binom{r}{m}\sum_j \binom{m-r+s}{j}\binom{r-m}{n-j} \\
&\quad= \binom{r}{m}\binom{s}{n}. && \text{from Eq. (21)}
\end{aligned}
$$

J. F. Plaff, *Nova Acta Acad. Scient. Petr.* **11** (1797), 38–57.

32. [*M20*] Show that $\sum_k {n \brack k} x^k = x^{\overline{n}}$, where $x^{\overline{n}}$ is the rising factorial power defined in Eq. 1.2.5-(19).

Proposition. *$\sum_k {n \brack k} x^k = x^{\overline{n}}$.*

Proof. Let $x^{\overline{n}}$ be the rising factorial power defined as

$$x^{\overline{n}} = \prod_{0 \le k \le n-1} (x+k).$$

We must show that

$$\sum_k {n \brack k} x^k = x^{\overline{n}}.$$

But

$$\begin{aligned}
\sum_k \left[{n \atop k}\right] x^k &= \sum_k \left[{n \atop k}\right] (-1)^k (-x)^k \\
&= \sum_k \left[{n \atop k}\right] (-1)^{-k} (-x)^k \\
&= \sum_k \left[{n \atop k}\right] (-1)^{n-n} (-1)^{-k} (-x)^k \\
&= \sum_k \left[{n \atop k}\right] (-1)^{n} (-1)^{n} (-1)^{-k} (-x)^k \\
&= (-1)^n \sum_k \left[{n \atop k}\right] (-1)^{n-k} (-x)^k \\
&= (-1)^n (-x)^{\underline{n}} && \text{from Eq. (44)} \\
&= x^{\overline{n}} && \text{from Eq. 1.2.5-(20)}
\end{aligned}$$

as we needed to show. □

33. [*M20*] (A. Vandermonde, 1772.) Show that the binomial formula is valid also when it involves factorial powers instead of the ordinary powers. In order words, prove that

$$(x+y)^{\underline{n}} = \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}; \qquad (x+y)^{\overline{n}} = \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}.$$

We may prove the formula for factorial falling.

Proposition. $(x+y)^{\underline{n}} = \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$.

Proof. Let n be an arbitrary nonnegative integer. We must show that

$$(x+y)^{\underline{n}} = \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}.$$

If $n = 0$,

$$\begin{aligned}
(x+y)^{\underline{0}} &= 1 \\
&= \binom{0}{0} x^{\underline{0}} y^{\underline{0}} \\
&= \sum_k \binom{0}{k} x^{\underline{k}} y^{\underline{0-k}}.
\end{aligned}$$

Then, assuming

$$(x+y)^{\underline{n}} = \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}$$

we must show that

$$(x+y)^{\underline{n+1}} = \sum_k \binom{n+1}{k} x^{\underline{k}} y^{\underline{n+1-k}}.$$

But

$$\begin{aligned}
(x+y)^{\underline{n+1}} &= (x+y-n)(x+y)^{\underline{n}} \\
&= (x+y-n)\sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}} \\
&= \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n-k}}((x-k)+(y-(n-k))) \\
&= \sum_k \binom{n}{k} x^{\underline{k+1}} y^{\underline{n-k}} + \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n+1-k}} \\
&= \sum_k \binom{n}{k-1} x^{\underline{k}} y^{\underline{n+1-k}} + \sum_k \binom{n}{k} x^{\underline{k}} y^{\underline{n+1-k}} \\
&= \sum_k \left(\binom{n}{k-1} + \binom{n}{k}\right) x^{\underline{k}} y^{\underline{n+1-k}} \\
&= \sum_k \binom{n+1}{k} x^{\underline{k}} y^{\underline{n+1-k}} && \text{from Eq. (9)}
\end{aligned}$$

as we needed to show. □

We may also prove the formula for factorial rising.

Proposition. $(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}$.

Proof. Let n be an arbitrary nonnegative integer. We must show that

$$(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}.$$

If $n=0$,

$$\begin{aligned}
(x+y)^{\overline{0}} &= 1 \\
&= \binom{0}{0} x^{\overline{0}} y^{\overline{0}} \\
&= \sum_k \binom{0}{k} x^{\overline{k}} y^{\overline{0-k}}.
\end{aligned}$$

Then, assuming

$$(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}$$

we must show that

$$(x+y)^{\overline{n+1}} = \sum_k \binom{n+1}{k} x^{\overline{k}} y^{\overline{n+1-k}}.$$

But

$$\begin{aligned}
(x+y)^{\overline{n+1}} &= (x+y+n)(x+y)^{\overline{n}} \\
&= (x+y+n)\sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}} \\
&= \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n-k}}((x+k)+(y+(n-k))) \\
&= \sum_k \binom{n}{k} x^{\overline{k+1}} y^{\overline{n-k}} + \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n+1-k}} \\
&= \sum_k \binom{n}{k-1} x^{\overline{k}} y^{\overline{n+1-k}} + \sum_k \binom{n}{k} x^{\overline{k}} y^{\overline{n+1-k}} \\
&= \sum_k \left(\binom{n}{k-1} + \binom{n}{k}\right) x^{\overline{k}} y^{\overline{n+1-k}} \\
&= \sum_k \binom{n+1}{k} x^{\overline{k}} y^{\overline{n+1-k}} && \text{from Eq. (9)}
\end{aligned}$$

as we needed to show. □

34. [*M23*] (*Torelli's sum.*) In the light of the previous exercise, show that Abel's generalization, Eq. (16), of the binomial formula is true also for rising powers:

$$(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x(x-kz+1)^{\overline{k-1}}(y+kz)^{\overline{n-k}}.$$

Proposition. $(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x(x-kz+1)^{\overline{k-1}}(y+kz)^{\overline{n-k}}$.

Proof. Let n be an arbitrary nonnegative integer and x an arbitrary nonzero real number. We must show that

$$(x+y)^{\overline{n}} = \sum_k \binom{n}{k} x(x-kz+1)^{\overline{k-1}}(y+kz)^{\overline{n-k}}.$$

Given the general identity for arbitrary a

$$\begin{aligned}
a^{\overline{n}} &= \frac{\Gamma(a+n)}{\Gamma(a)} \\
&= \frac{(a+n)!a}{(a+n)a!} \\
&= \frac{(a+n-1)!}{(a-1)!} \\
&= \frac{(a+n-1)!}{(a+n-1-n)!} \\
&= \frac{n!(a+n-1)!}{n!(a+n-1-n)!} \\
&= \frac{n!(a+n-1)!}{n!(a+n-1-n)!} \\
&= n!\binom{a+n-1}{n},
\end{aligned}$$

we have that

$$\begin{aligned}
&(x+y)^{\overline{n}} \\
&\quad = n!\binom{x+y+n-1}{n} \\
&\quad = n!\sum_k \binom{x-(z-1)k}{k}\binom{y+nz-1-(z-1)(n-k)}{n-k}\frac{x}{x-(z-1)k} && \text{from Eq. (26)} \\
&\quad = n!\sum_k \frac{x}{k}\frac{k}{x-(z-1)k}\binom{x-(z-1)k}{k}\binom{y+nz-1-(z-1)(n-k)}{n-k} \\
&\quad = n!\sum_k \frac{x}{k}\binom{x-(z-1)k-1}{k-1}\binom{y+nz-1-(z-1)(n-k)}{n-k} && \text{from Eq. (7)} \\
&\quad = n!\sum_k \frac{x}{k}\binom{x-(z-1)k-1}{k-1}\binom{y+kz+n-k-1}{n-k} \\
&\quad = \sum_k \frac{n!}{k!(n-k)!}(k-1)!(n-k)!x\binom{x-k(z-1)-1}{k-1}\binom{y+kz+n-k-1}{n-k} \\
&\quad = \sum_k \binom{n}{k}x(k-1)!\binom{x-kz+1+k-1-1}{k-1}(n-k)!\binom{y+kz+n-k-1}{n-k} \\
&\quad = \sum_k \binom{n}{k}x(x-kz+1)^{\overline{k-1}}(y+kz)^{\overline{n-k}}
\end{aligned}$$

as we needed to show. □

A. Vandermonde, *Mém. Acad. Roy. Sci.* (Paris, 1772), part 1, 492; C. Kramp, *Élémens d'Arithmétique Universelle* (Cologne: 1808), 359; G. Torelli, *Giornale di Mat. Battaglini* **33** (1895), 179–182; H. A. Rothe, *Formulæde Serierum Reversione* (Leipzig: 1793), 18.

35. [*M23*] Prove the addition formulas (46) for Stirling numbers directly from the definitions, Eqs. (44) and (45).

We may prove the addition formula for Stirling numbers of the first kind.

Proposition. $\left[{n+1 \atop m}\right] = n\left[{n \atop m}\right] + \left[{n \atop m-1}\right]$.

Proof. Let m and n be arbitrary nonnegative integers. We must show that

$$\begin{bmatrix} n+1 \\ m \end{bmatrix} = n\begin{bmatrix} n \\ m \end{bmatrix} + \begin{bmatrix} n \\ m-1 \end{bmatrix}.$$

But from Eq. (44),

$$\begin{aligned}
\sum_k (-1)^{n+1-k} \left[{n+1 \atop k}\right] x^k &= x^{\underline{n+1}} \\
&= (x-n)x^{\underline{n}} \\
&= -nx^{\underline{n}} + xx^{\underline{n}} \\
&= -n\sum_k (-1)^{n-k} \left[{n \atop k}\right] x^k + x\sum_k (-1)^{n-k} \left[{n \atop k}\right] x^k \\
&= -n\sum_k (-1)^{n-k} \left[{n \atop k}\right] x^k + x\sum_k (-1)^{n-(k-1)} \left[{n \atop k-1}\right] x^{k-1} \\
&= \sum_k (-1)^{n+1-k} n \left[{n \atop k}\right] x^k + \sum_k (-1)^{n+1-k} \left[{n \atop k-1}\right] x^k \\
&= \sum_k (-1)^{n+1-k} \left(n\left[{n \atop k}\right] + \left[{n \atop k-1}\right] \right) x^k
\end{aligned}$$

and hence the result equating coefficients. □

We may also prove the addition formula for Stirling numbers of the second kind.

Proposition. $\left\{{n+1 \atop m}\right\} = m\left\{{n \atop m}\right\} + \left\{{n \atop m-1}\right\}$.

Proof. Let m and n be arbitrary nonnegative integers. We must show that

$$\left\{{n+1 \atop m}\right\} = m\left\{{n \atop m}\right\} + \left\{{n \atop m-1}\right\}.$$

But from Eq. (45),

$$\begin{aligned}
\sum_k \left\{{n+1 \atop k}\right\} x^{\underline{k}} &= x^{n+1} \\
&= xx^n \\
&= x\sum_k \left\{{n \atop k}\right\} x^{\underline{k}} \\
&= \sum_k \left\{{n \atop k}\right\} xx^{\underline{k}} \\
&= \sum_k \left\{{n \atop k}\right\} \left(xx^{\underline{k}} + kx^{\underline{k}} - kx^{\underline{k}}\right) \\
&= \sum_k \left\{{n \atop k}\right\} \left((x-k)x^{\underline{k}} + kx^{\underline{k}}\right) \\
&= \sum_k \left\{{n \atop k}\right\} \left(x^{\underline{k+1}} + kx^{\underline{k}}\right) \\
&= \sum_k \left\{{n \atop k}\right\} kx^{\underline{k}} + \sum_k \left\{{n \atop k}\right\} x^{\underline{k+1}} \\
&= \sum_k k\left\{{n \atop k}\right\} x^{\underline{k}} + \sum_k \left\{{n \atop k-1}\right\} x^{\underline{k}} \\
&= \sum_k \left(k\left\{{n \atop k}\right\} + \left\{{n \atop k-1}\right\} \right) x^{\underline{k}}
\end{aligned}$$

and hence the result equating coefficients. □

36. [*M10*] What is the sum $\sum_k \binom{n}{k}$ of the numbers in each row of Pascal's triangle? What is the sum of these numbers with alternating signs, $\sum_k \binom{n}{k}(-1)^k$?

Assuming n a nonnegative integer, from Eq. (13),

$$\begin{aligned}\sum_k \binom{n}{k} &= \sum_k \binom{n}{k} 1^k 1^{n-k}\\ &= (1+1)^n\\ &= 2^n,\end{aligned}$$

and

$$\begin{aligned}\sum_k \binom{n}{k}(-1)^k &= \sum_k \binom{n}{k}(-1)^k 1^{n-k}\\ &= (-1+1)^n\\ &= 0^n\\ &= \delta_{n0}.\end{aligned}$$

37. [*M10*] From the answers to the preceding exercise, deduce the value of the sum of every other entry in a row, $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots$.

Again assuming n a nonnegative integer, from the preceding exercise,

$$\begin{aligned}\sum_{k \text{ even}} \binom{n}{k} &= \sum_k \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k}\\ &= \frac{\sum_k \binom{n}{k} + \sum_{k \text{ even}} \binom{n}{k} - \sum_{k \text{ odd}} \binom{n}{k}}{2}\\ &= \frac{\sum_k \binom{n}{k} + \sum_{k \text{ even}} \binom{n}{k}(-1)^k + \sum_{k \text{ odd}} \binom{n}{k}(-1)^k}{2}\\ &= \frac{\sum_k \binom{n}{k} + \sum_k \binom{n}{k}(-1)^k}{2}\\ &= \frac{2^n + \delta_{n0}}{2}\\ &= \begin{cases} 1 & \text{if } n = 0\\ 2^{n-1} & \text{if } n > 0.\end{cases}\end{aligned}$$

38. [*HM30*] (C. Ramus, 1834.) Generalizing the result of the preceding exercise, show that we have the following formula, given that $0 \le k < m$:

$$\binom{n}{k} + \binom{n}{m+k} + \binom{n}{2m+k} + \cdots = \frac{1}{m} \sum_{0 \le j < m} \left(2 \cos \frac{j\pi}{m}\right)^n \cos \frac{j(n-2k)\pi}{m}.$$

For example,

$$\binom{n}{1} + \binom{n}{4} + \binom{n}{7} + \cdots = \frac{1}{3}\left(2^n + 2 \cos \frac{(n-2)\pi}{3}\right).$$

[*Hint:* Find the right combinations of these coefficients multiplied by mth roots of unity.] This identity is particularly remarkable when $m \ge n$.

Proposition. $\sum_{j \ge 0} \binom{n}{jm+k} = \frac{1}{m} \sum_{0 \le j < m} \left(2 \cos \frac{j\pi}{m}\right)^n \cos \frac{j(n-2k)\pi}{m}$.

Proof. Let k and m be arbitrary integers such that $0 \leq k < m$. We must show that

$$\sum_{j \geq 0} \binom{n}{jm+k} = \frac{1}{m} \sum_{0 \leq j < m} \left(2 \cos \frac{j\pi}{m}\right)^n \cos \frac{j(n-2k)\pi}{m}.$$

Given $\omega = e^{2\pi i/m}$ and the sum of the geometric progression for t restricted such that $t \bmod m = k$,

$$\begin{aligned}
\sum_{0 \leq j < m} \omega^{j(t-k)} &= \sum_{0 \leq j < m} \left(\omega^{t-k}\right)^j \\
&= \sum_{0 \leq j < m} \left(e^{2\pi i(t-k)/m}\right)^j \\
&= \sum_{0 \leq j < m} \left((-1)^{2(t-k)/m}\right)^j \\
&= \sum_{0 \leq j < m} \left(1^{(t-k)/m}\right)^j \\
&= [t - k \equiv 0 \pmod{m}] \sum_{0 \leq j < m} 1^j \\
&= [t - k \equiv 0 \pmod{m}] m
\end{aligned}$$

we have for the real part that

$$\begin{aligned}
\sum_{j\geq 0}\binom{n}{jm+k} &= \sum_{t \bmod m = k}\binom{n}{t} \\
&= \sum_{t-k\equiv 0 \pmod m}\binom{n}{t} \\
&= \sum_{t}\binom{n}{t}[t-k\equiv 0 \pmod m] \\
&= \sum_{t}\binom{n}{t}\frac{1}{m}\sum_{0\leq j<m}\omega^{j(t-k)} \\
&= \frac{1}{m}\sum_{0\leq j<m}\omega^{-jk}\sum_{t}\binom{n}{t}\omega^{jt} \\
&= \frac{1}{m}\sum_{0\leq j<m}\omega^{-jk}\left(1+\omega^{j}\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}\omega^{-jk}\left(\omega^{j}+1\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}\omega^{-jk}\left(\omega^{j/2}\omega^{j/2}+\omega^{j/2}\omega^{-j/2}\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}\omega^{-jk}\omega^{jn/2}\left(\omega^{j/2}+\omega^{-j/2}\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}\omega^{j(n/2-k)}\left(\omega^{j/2}+\omega^{-j/2}\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}e^{2\pi ij(n/2-k)/m}\left(e^{2\pi ij/m2}+e^{-2\pi ij/m2}\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}e^{i\frac{j(n-2k)\pi}{m}}\left(e^{i\frac{j\pi}{m}}+e^{-i\frac{j\pi}{m}}\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}\cos\frac{j(n-2k)\pi}{m}\left(2\cos\frac{j\pi}{m}\right)^{n} \\
&= \frac{1}{m}\sum_{0\leq j<m}\left(2\cos\frac{j\pi}{m}\right)^{n}\cos\frac{j(n-2k)\pi}{m}
\end{aligned}$$

as we needed to show. □

C. Ramus, *Crelle* **11** (1834), 353–355; *CMath*, exercises 5.75 and 6.57.

39. [*M10*] What is the sum $\sum_k \left[{n \atop k}\right]$ of the numbers in each row of Stirling's first triangle? What is the sum of these numbers with alternating signs? (See exercise 36.)

Assuming n a nonnegative integer, from exercise 32 we have that

$$\begin{aligned}
\sum_k \begin{bmatrix} n \\ k \end{bmatrix} &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} 1^k \\
&= 1^{\overline{n}} \\
&= \prod_{0 \leq j \leq n-1} (1+j) && \text{from Eq. 1.2.5-(19)} \\
&= \prod_{1 \leq j \leq n} j \\
&= n!
\end{aligned}$$

and from Eq. (44) we have that

$$\begin{aligned}
\sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k &= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{-k} \\
&= \sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k-n} \\
&= \sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{-n} \\
&= \sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n} \\
&= \sum_k (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} 1^k (-1)^{n} \\
&= 1^{\underline{n}} (-1)^n \\
&= \prod_{0 \leq j \leq n-1} (1-j)(-1)^n && \text{from Eq. 1.2.5-(18)} \\
&= \begin{cases} 1 & \text{if } n = 0 \\ -1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_{n0} - \delta_{n1}.
\end{aligned}$$

40. [*HM17*] The *beta function* $B(x,y)$ is defined for positive real numbers x, y by the formula $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$.

a) Show that $B(x,1) = B(1,x) = 1/x$.

b) Show that $B(x+1,y) + B(x,y+1) = B(x,y)$.

c) Show that $B(x,y) = ((x+y)/y)B(x,y+1)$.

Let x and y be arbitrary positive real numbers and define the *beta* function $B(x,y)$ as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

a) We may show the first identity.

Proposition. $B(x,1) = B(1,x) = 1/x$.

Proof. We must show that

$$B(x,1) = B(1,x) = 1/x. \tag{68}$$

But by definition, both

$$\begin{aligned} B(x,1) &= \int_0^1 t^{x-1}(1-t)^{1-1}dt \\ &= -\int_1^0 (1-t)^{x-1}(1-(1-t))^{1-1}dt \\ &= \int_0^1 t^{1-1}(1-t)^{x-1}dt \\ &= B(1,x) \end{aligned}$$

and

$$\begin{aligned} B(x,1) &= \int_0^1 t^{x-1}(1-t)^{1-1}dt \\ &= \int_0^1 t^{x-1}dt \\ &= \left.\frac{t^x}{x}\right|_0^1 \\ &= \frac{1}{x} - \frac{0}{x} \\ &= \frac{1}{x} \end{aligned}$$

and hence the result. ☐

b) We may show the second identity.

Proposition. $B(x+1,y) + B(x,y+1) = B(x,y)$.

Proof. We must show that

$$B(x+1,y) + B(x,y+1) = B(x,y). \tag{71}$$

But

$$\begin{aligned} B(x+1,y) + B(x,y+1) &= \int_0^1 t^{x+1-1}(1-t)^{y-1}dt + \int_0^1 t^{x-1}(1-t)^{y+1-1}dt \\ &= \int_0^1 \left(t^{x+1-1}(1-t)^{y-1} + t^{x-1}(1-t)^{y+1-1}\right) dt \\ &= \int_0^1 \left(tt^{x-1}(1-t)^{y-1} + (1-t)t^{x-1}(1-t)^{y-1}\right) dt \\ &= \int_0^1 (t+1-t)t^{x-1}(1-t)^{y-1}dt \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}dt \\ &= B(x,y) \end{aligned}$$

and hence the result. ☐

c) We may show the third identity.

Proposition. $B(x,y) = ((x+y)/y)B(x,y+1)$.

Proof. We must show that

$$B(x,y) = ((x+y)/y)B(x,y+1). \tag{73}$$

But from integration by parts

$$-\int_0^1 t^x y(1-t)^{y-1}dt = t^x(1-t)^y|_0^1 - \int_0^1 xt^{x-1}(1-t)^y dt$$

which gives us that

$$\begin{aligned}
B(x+1,y) &= \int_0^1 t^x(1-t)^{y-1}dt \\
&= -\left.\frac{t^x(1-t)^y}{y}\right|_0^1 + \frac{x}{y}\int_0^1 t^{x-1}(1-t)^y dt \\
&= -\left.\frac{t^x(1-t)^y}{y}\right|_0^1 + \frac{x}{y}B(x,y+1) \\
&= -\frac{1^x(1-1)^y}{y} + \frac{0^x(1-0)^y}{y} + \frac{x}{y}B(x,y+1) \\
&= \frac{x}{y}B(x,y+1).
\end{aligned}$$

Then, from (b)

$$\begin{aligned}
B(x,y) &= B(x+1,y) + B(x,y+1) \\
&= \frac{x}{y}B(x,y+1) + B(x,y+1) \\
&= \frac{x+y}{y}B(x,y+1)
\end{aligned}$$

and hence the result. □

41. [*HM22*] We proved a relation between the gamma function and the beta function in exercise 1.2.5-19, by showing that $\Gamma_m(x) = m^x B(x,m+1)$, if m is a positive integer.

a) Prove that

$$B(x,y) = \frac{\Gamma_m(y)m^x}{\Gamma_m(x+y)}B(x,y+m+1).$$

b) Show that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Define the *gamma* function for positive integers m as

$$\Gamma_m(x) = \frac{m^x m!}{\prod_{0\le j\le m}(x+j)},$$

so that from exercise 1.2.5-19

$$\Gamma_m(x) = m^x B(x,m+1).$$

a) We may prove the first identity.

Proposition. $B(x,y) = \frac{\Gamma_m(y)m^x}{\Gamma_m(x+y)}B(x,y+m+1)$.

Proof. Let m be a positive integer. We must show that

$$B(x,y) = \frac{\Gamma_m(y)m^x}{\Gamma_m(x+y)}B(x,y+m+1).$$

As an initial corollary, we will first show for positive integers k that

$$B(x,y) = \prod_{0\le j<k} \frac{x+y+j}{y+j}B(x,y+k).$$

If $k=1$ we have from exercise 40 (c) that

$$\begin{aligned} B(x,y) &= \frac{x+y}{y}B(x,y+1) \\ &= \prod_{0\le j<k} \frac{x+y+j}{y+j}B(x,y+k). \end{aligned}$$

Then, assuming

$$B(x,y) = \prod_{0\le j<k} \frac{x+y+j}{y+j}B(x,y+k)$$

we must show that

$$B(x,y) = \prod_{0\le j<k+1} \frac{x+y+j}{y+j}B(x,y+k+1).$$

But again from exercise 40 (c)

$$\begin{aligned} B(x,y) &= \prod_{0\le j<k} \frac{x+y+j}{y+j}B(x,y+k) \\ &= \prod_{0\le j<k} \frac{x+y+j}{y+j}\frac{x+y+k}{y+k}B(x,y+k+1) \\ &= \prod_{0\le j<k+1} \frac{x+y+j}{y+j}B(x,y+k+1) \end{aligned}$$

and hence the interim result. Then finally,

$$\begin{aligned} B(x,y) &= \prod_{0\le j<m+1} \frac{x+y+j}{y+j}B(x,y+m+1) \\ &= \prod_{0\le j\le m} \frac{x+y+j}{y+j}B(x,y+m+1) \\ &= \frac{\prod_{0\le j\le m}(x+y+j)}{\prod_{0\le j\le m}(y+j)}B(x,y+m+1) \\ &= \frac{m^{x+y}m!\prod_{0\le j\le m}(x+y+j)}{m^{x+y}m!\prod_{0\le j\le m}(y+j)}B(x,y+m+1) \\ &= \left(\frac{m^y m!}{\prod_{0\le j\le m}(y+j)}\middle/\frac{m^{x+y}m!}{\prod_{0\le j\le m}(x+y+j)}\right)m^x B(x,y+m+1) \\ &= \frac{\Gamma_m(y)m^x}{\Gamma_m(x+y)}B(x,y+m+1) \end{aligned}$$

as we needed to show. ∎

b) We may show the second identity.

Proposition. $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Proof. Let m be a positive integer. We must show that

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

It is sufficient to show that

$$\lim_{m\to\infty} m^x B(x, y+m+1) = \Gamma(x).$$

Note that B is monotonically decreasing for positive x and y. That is, if $(x,y) \leq (x, y+1)$, then $B(x,y) \geq B(x,y+1)$, since from exercise 40

$$\begin{aligned}
& x > 0 \wedge y > 0 \\
\iff\quad & \frac{x}{y} \geq 0 \\
\iff\quad & \frac{x}{y} + 1 \geq 1 \\
\iff\quad & \frac{x+y}{y} \geq 1 \\
\iff\quad & \frac{x+y}{y} B(x,y+1) \geq B(x,y+1) \\
\iff\quad & B(x,y) \geq B(x,y+1).
\end{aligned}$$

Then, since B is monotonically decreasing and from exercise 1.2.5-19,

$$\begin{aligned}
& y+m+1 \leq y+m+1 < n+m+2 \\
\iff\quad & B(x, y+m+2) < B(x, y+m+1) \leq B(x, y+m+1) \\
\iff\quad & \frac{\Gamma_{y+m+1}(x)}{(y+m+1)^x} < B(x,y+m+1) \leq \frac{\Gamma_{y+m}(x)}{(y+m)^x} \\
\iff\quad & \frac{\Gamma_{y+m+1}(x)}{m^x(1+(y+1)/m)^x} < B(x,y+m+1) \leq \frac{\Gamma_{y+m}(x)}{m^x(1+y/m)^x} \\
\iff\quad & \left(\frac{m}{y+m+1}\right)^x \Gamma_{y+m+1}(x) < m^x B(x,y+m+1) \\
& \leq \left(\frac{m}{y+m}\right)^x \Gamma_{y+m}(x) \\
\iff\quad & \lim_{m\to\infty} \left(\frac{m}{y+m+1}\right)^x \Gamma_{y+m+1}(x) < \lim_{m\to\infty} m^x B(x,y+m+1) \\
& \leq \lim_{m\to\infty} \left(\frac{m}{y+m}\right)^x \Gamma_{y+m}(x) \\
\iff\quad & \lim_{m\to\infty} \Gamma_{y+m+1}(x) < \lim_{m\to\infty} m^x B(x,y+m+1) \leq \lim_{m\to\infty} \Gamma_{y+m}(x) \\
\iff\quad & \lim_{m\to\infty} \Gamma_m(x) < \lim_{m\to\infty} m^x B(x,y+m+1) \leq \lim_{m\to\infty} \Gamma_m(x) \\
\iff\quad & \Gamma(x) < \lim_{m\to\infty} m^x B(x,y+m+1) \leq \Gamma(x) \\
\iff\quad & \lim_{m\to\infty} m^x B(x,y+m+1) = \Gamma(x).
\end{aligned}$$

And so from (a),

$$\begin{aligned} B(x,y) &= \lim_{m\to\infty} \frac{\Gamma_m(y)m^x}{\Gamma_m(x+y)} B(x, y+m+1) \\ &= \lim_{m\to\infty} \frac{m^x B(x,y+m+1)\Gamma_m(y)}{\Gamma_m(x+y)} \\ &= \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \end{aligned}$$

as we needed to show. □

42. [*HM10*] Express the binomial coefficient $\binom{r}{k}$ in terms of the beta function defined above. (This gives us a way to extend the definition to all real values of k.)

From exercise 41 (b) we have

$$\begin{aligned} \binom{r}{k} &= \frac{r!}{k!(r-k)!} \\ &= \frac{(r+2)!(k+1)(r-k+1)}{(r+1)(r+2)(k+1)!(r-k+1)!} \\ &= \frac{\Gamma(r+2)}{(r+1)\Gamma(k+1)\Gamma(r-k+1)} \\ &= \frac{\Gamma(k+1+r-k+1)}{(r+1)\Gamma(k+1)\Gamma(r-k+1)} \\ &= \frac{1}{(r+1)B(k+1,r-k+1)}. \end{aligned}$$

L. Ramshaw, *Inf. Proc. Letters* **6** (1977), 223–226.

43. [*HM20*] Show that $B(1/2, 1/2) = \pi$. (From exercise 41 we may now conclude that $\Gamma(1/2) = \sqrt{\pi}$.)

Proposition. $B(1/2, 1/2) = \pi$.

Proof. Define the *beta* function $B(x, y)$ as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

We must show that

$$B(1/2, 1/2) = \pi.$$

But for $u = t^{1/2}$, $du = \frac{1}{2t^{1/2}}dt$,

$$\begin{aligned}
B(1/2, 1/2) &= \int_0^1 t^{1/2-1}(1-t)^{1/2-1}dt \\
&= \int_0^1 \frac{1}{t^{1/2}(1-t)^{1/2}}dt \\
&= \int_0^1 \frac{2t^{1/2}}{t^{1/2}(1-u^2)^{1/2}}du \\
&= 2\int_0^1 \frac{1}{(1-u^2)^{1/2}}du \\
&= 2 \arcsin u|_0^1 \\
&= 2(\pi/2 - 0) \\
&= \pi
\end{aligned}$$

as we needed to show. □

44. [*HM20*] Using the generalized binomial coefficient suggested in exercise 42, show that

$$\binom{r}{1/2} = 2^{2r+1} \bigg/ \binom{2r}{r}\pi.$$

Proposition. $\binom{r}{1/2} = 2^{2r+1} \Big/ \binom{2r}{r}\pi.$

Proof. As a corollary, we will prove Gauss's multiplication formula. That is, that

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{0\le k\le n-1} \Gamma\left(z + \frac{k}{n}\right).$$

By Stirling's formula as $m \to \infty$, we have

$$\begin{aligned}
\Gamma\left(z + \frac{k}{n}\right) &= \left(z + \frac{k}{n} - 1\right)\Gamma\left(z + \frac{k}{n} - 1\right) \\
&= \lim_{m\to\infty} m^{z+k/n-1} m! \bigg/ \prod_{0\le j\le m-1}\left(z + \frac{k}{n} + j\right) \\
&= \lim_{m\to\infty} m^{z+k/n-1}\sqrt{2\pi m}\left(\frac{m}{e}\right)^m \bigg/ \prod_{0\le j\le m-1}\left(z + \frac{k}{n} + j\right) \\
&= \lim_{m\to\infty} m^{z+k/n-1/2}\sqrt{2\pi}\left(\frac{mn}{e}\right)^m \bigg/ \prod_{0\le j\le m-1}(nz + k + jn),
\end{aligned}$$

and

$$
\begin{aligned}
&\prod_{0\le k\le n-1} \Gamma\left(z+\frac{k}{n}\right)\\
&\quad= \lim_{m\to\infty} m^{nz-n/2} m^{\sum_{0\le k\le n-1} k/n} \left(\sqrt{2\pi}\right)^n \left(\frac{mn}{e}\right)^{mn} \Bigg/ \prod_{0\le k\le n-1}\ \prod_{0\le j\le m-1} (nz+k+jn)\\
&\quad= \lim_{m\to\infty} m^{nz-n/2} m^{\sum_{0\le k\le n-1} k/n} \left(\sqrt{2\pi}\right)^n \left(\frac{mn}{e}\right)^{mn} \Bigg/ \prod_{0\le j\le mn-1} (nz+j)\\
&\quad= \lim_{m\to\infty} m^{nz-1/2} \left(\sqrt{2\pi}\right)^n \left(\frac{mn}{e}\right)^{mn} \Bigg/ \prod_{0\le j\le mn-1} (nz+j)\\
&\quad= \lim_{m\to\infty} (m/n)^{nz-1/2} \left(\sqrt{2\pi}\right)^n \left(\frac{(m/n)n}{e}\right)^{(m/n)n} \Bigg/ \prod_{0\le j\le (m/n)n-1} (nz+j)\\
&\quad= \lim_{m\to\infty} m^{nz-1/2} n^{1/2-nz} \left(\sqrt{2\pi}\right)^n \left(\frac{m}{e}\right)^{m} \Bigg/ \prod_{0\le j\le m-1} (nz+j)\\
&\quad= \lim_{m\to\infty} m^{nz-1} n^{1/2-nz} \left(\sqrt{2\pi}\right)^{n-1} \sqrt{2\pi m} \left(\frac{m}{e}\right)^{m} \Bigg/ \prod_{0\le j\le m-1} (nz+j)\\
&\quad= \lim_{m\to\infty} m^{nz-1} n^{1/2-nz} \left(\sqrt{2\pi}\right)^{n-1} m! \Bigg/ \prod_{0\le j\le m-1} (nz+j)\\
&\quad= (2\pi)^{(n-1)/2} n^{1/2-nz} \lim_{m\to\infty} m^{nz-1} m! \Bigg/ \prod_{0\le j\le m-1} (nz+j)\\
&\quad= (2\pi)^{(n-1)/2} n^{1/2-nz} (nz-1)\Gamma(nz-1)\\
&\quad= (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz),
\end{aligned}
$$

and hence the result

$$\Gamma(nz) = (2\pi)^{(1-n)/2} n^{nz-1/2} \prod_{0\le k\le n-1} \Gamma\left(z+\frac{k}{n}\right).$$

For the proof, we must show that

$$\binom{r}{1/2} = 2^{2r+1} \Bigg/ \binom{2r}{r}\pi.$$

But from exercise 42 and since $\Gamma(3/2) = \Gamma(1/2)/2 = \sqrt{\pi}/2$,

$$
\begin{aligned}
\binom{r}{1/2} &= \frac{1}{(r+1)B(1/2+1, r-1/2+1)}\\
&= \frac{1}{(r+1)B(3/2, r+1/2)}\\
&= \frac{\Gamma(3/2+r+1/2)}{(r+1)\Gamma(3/2)\Gamma(r+1/2)}\\
&= \frac{\Gamma(r+2)}{(r+1)\Gamma(3/2)\Gamma(r+1/2)}\\
&= \frac{2\Gamma(r+2)}{(r+1)\sqrt{\pi}\Gamma(r+1/2)}
\end{aligned}
$$

and

$$\begin{aligned}\binom{2r}{r} &= \frac{1}{(2r+1)B(r+1, 2r-r+1)} \\ &= \frac{1}{(2r+1)B(r+1, r+1)} \\ &= \frac{\Gamma(r+1+r+1)}{(2r+1)\Gamma(r+1)\Gamma(r+1)} \\ &= \frac{\Gamma(2r+2)}{(2r+1)\Gamma(r+1)\Gamma(r+1)}.\end{aligned}$$

Multiplying both equalities yields

$$\begin{aligned}\binom{r}{1/2}\binom{2r}{r} &= \frac{2\Gamma(r+2)}{(r+1)\sqrt{\pi}\Gamma(r+1/2)} \frac{\Gamma(2r+2)}{(2r+1)\Gamma(r+1)\Gamma(r+1)} \\ &= \frac{2\Gamma(r+2)\Gamma(2r+2)}{(r+1)\sqrt{\pi}\Gamma(r+1/2)(2r+1)\Gamma(r+1)\Gamma(r+1)} \\ &= \frac{2\Gamma(r+2)(2r+1)\Gamma(2r+1)}{(r+1)\sqrt{\pi}\Gamma(r+1/2)(2r+1)\Gamma(r+1)\Gamma(r+1)} \\ &= \frac{2\Gamma(r+2)\Gamma(2r+1)}{(r+1)\sqrt{\pi}\Gamma(r+1/2)\Gamma(r+1)\Gamma(r+1)} \\ &= \frac{2(r+1)\Gamma(r+1)\Gamma(2r+1)}{(r+1)\sqrt{\pi}\Gamma(r+1/2)\Gamma(r+1)\Gamma(r+1)} \\ &= \frac{2\Gamma(2r+1)}{\sqrt{\pi}\Gamma(r+1/2)\Gamma(r+1)}\end{aligned}$$

if and only if

$$\binom{r}{1/2} = 2\Gamma(2r+1) \Big/ \sqrt{\pi}\Gamma(r+1/2)\Gamma(r+1)\binom{2r}{r}.$$

That is, it is sufficient to show that

$$\frac{\Gamma(2r+1)}{\Gamma(r+1)\Gamma(r+1/2)} = \frac{2^{2r}}{\sqrt{\pi}}.$$

But, by Guass's multiplication formula,

$$\begin{aligned}&\frac{\Gamma(2r+1)}{\Gamma(r+1)\Gamma(r+1/2)} \\ &\quad= \frac{2r}{\Gamma(r+1)\Gamma(r+1/2)}\Gamma(2r) \\ &\quad= \frac{2r}{\Gamma(r+1)\Gamma(r+1/2)}(2\pi)^{(1-2)/2}2^{2r-1/2}\prod_{0\le k\le 2-1}\Gamma\left(r+\frac{k}{2}\right) \\ &\quad= \frac{2r}{\Gamma(r+1)\Gamma(r+1/2)}(2\pi)^{-1/2}2^{2r-1/2}\Gamma(r)\Gamma\left(r+\frac{1}{2}\right) \\ &\quad= \frac{2^{2r}r\Gamma(r)\Gamma(r+1/2)}{\sqrt{\pi}\Gamma(r+1)\Gamma(r+1/2)} \\ &\quad= \frac{2^{2r}r\Gamma(r)}{\sqrt{\pi}r\Gamma(r)} \\ &\quad= \frac{2^{2r}}{\sqrt{\pi}}.\end{aligned}$$

And so,

$$\begin{aligned}\binom{r}{1/2} &= 2\Gamma(2r+1)\Big/\sqrt{\pi}\Gamma(r+1/2)\Gamma(r+1)\binom{2r}{r} \\ &= \frac{\Gamma(2r+1)}{\Gamma(r+1)\Gamma(r+1/2)}2\Big/\sqrt{\pi}\binom{2r}{r} \\ &= \frac{2^{2r}}{\sqrt{\pi}}2\Big/\sqrt{\pi}\binom{2r}{r} \\ &= 2^{2r}2\Big/\sqrt{\pi}\sqrt{\pi}\binom{2r}{r} \\ &= 2^{2r+1}\Big/\binom{2r}{r}\pi\end{aligned}$$

as we needed to show. □

45. [*HM21*] Using the generalized binomial coefficient suggested in exercise 42, find $\lim_{r\to\infty}\binom{r}{k}/r^k$.

From exercise 42 and Stirling's approximation,

$$\begin{aligned}\lim_{r\to\infty}\binom{r}{k}/r^k &= \lim_{r\to\infty}\frac{1}{r^k(r+1)B(k+1,r-k+1)} \\ &= \lim_{r\to\infty}\frac{\Gamma(k+1+r-k+1)}{r^k(r+1)\Gamma(k+1)\Gamma(r-k+1)} \\ &= \lim_{r\to\infty}\frac{\Gamma(r+2)}{r^k(r+1)\Gamma(k+1)\Gamma(r-k+1)} \\ &= \frac{1}{\Gamma(k+1)}\lim_{r\to\infty}\frac{(r+1)\Gamma(r+1)}{r^k(r+1)\Gamma(r-k+1)} \\ &= \frac{1}{\Gamma(k+1)}\lim_{r\to\infty}\frac{\Gamma(r+1)}{r^k\Gamma(r-k+1)} \\ &= \frac{1}{\Gamma(k+1)}\lim_{r\to\infty}\frac{r!}{r^k(r-k)!} \\ &= \frac{1}{\Gamma(k+1)}\lim_{r\to\infty}\frac{\sqrt{2\pi r}(r/e)^r}{r^k\sqrt{2\pi(r-k)}((r-k)/e)^{r-k}} \\ &= \frac{1}{\Gamma(k+1)}\lim_{r\to\infty}\sqrt{\frac{r}{r-k}}\frac{1}{e^r}\frac{e^r}{e^k}\frac{(r-k)^k r^r}{r^k(r-k)^r} \\ &= \frac{1}{\Gamma(k+1)}\lim_{r\to\infty}\sqrt{\frac{r}{r-k}}\frac{1}{e^k}\frac{((r-k)/r)^k}{((r-k)/r)^r} \\ &= \frac{1}{\Gamma(k+1)}\lim_{r\to\infty}\sqrt{\frac{r}{r-k}}\frac{1}{e^k}\frac{(1-k/r)^k}{(1-k/r)^r} \\ &= \frac{1}{\Gamma(k+1)}\frac{1}{e^k}\lim_{r\to\infty}\frac{1}{(1-k/r)^r} \\ &= \frac{1}{\Gamma(k+1)}\frac{1}{e^k}\frac{1}{e^{-k}} \\ &= \frac{1}{\Gamma(k+1)}.\end{aligned}$$

▶ **46.** [*M21*] Using Stirling's approximation, Eq. 1.2.5-(7), find an approximate value of $\binom{x+y}{y}$, assuming that both x and y are large. In particular, find the approximate size of $\binom{2n}{n}$ when n is large.

Assuming that both x and y are large, using Stirling's approximation, we find that

$$\begin{aligned}
\binom{x+y}{y} &= \frac{1}{(x+y+1)B(x+1,y+1)} \\
&= \frac{1}{(x+y+1)B(y+1,x+y-y+1)} \\
&= \frac{1}{(x+y+1)B(y+1,x+1)} \\
&= \frac{\Gamma(x+y+2)}{(x+y+1)\Gamma(y+1)\Gamma(x+1)} \\
&= \frac{(x+y+1)\Gamma(x+y+1)}{(x+y+1)\Gamma(x+1)\Gamma(y+1)} \\
&= \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \\
&= \frac{\Gamma(x+y+1)}{\Gamma(x+1)\Gamma(y+1)} \\
&= \frac{(x+y)!}{x!y!} \\
&\approx \frac{\sqrt{2\pi(x+y)}((x+y)/e)^{x+y}}{\sqrt{2\pi x}(x/e)^x\sqrt{2\pi y}(y/e)^y} \\
&= \sqrt{\frac{x+y}{2\pi xy}}\frac{(x+y)^x(x+y)^y e^x e^y}{x^x y^y e^{x+y}} \\
&= \sqrt{\frac{1}{2\pi}\left(\frac{1}{x}+\frac{1}{y}\right)}\left(1+\frac{y}{x}\right)^x\left(1+\frac{x}{y}\right)^y.
\end{aligned}$$

In particular, when n is large,

$$\begin{aligned}
\binom{2n}{n} &= \binom{n+n}{n} \\
&\approx \sqrt{\frac{1}{2\pi}\left(\frac{1}{n}+\frac{1}{n}\right)}\left(1+\frac{n}{n}\right)^n\left(1+\frac{n}{n}\right)^n \\
&= \sqrt{\frac{1}{2\pi}\left(\frac{2}{n}\right)}(2)^n(2)^n \\
&= \frac{2^{2n}}{\sqrt{\pi n}} \\
&= \frac{4^n}{\sqrt{\pi n}}.
\end{aligned}$$

47. [*M21*] Given that k is an integer, show that

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{k}\binom{2r-k}{k}\Big/4^k = \binom{2r}{2k}\binom{2k}{k}\Big/4^k.$$

Give a simpler formula for the special case $r=-1/2$.

We may prove the equalities.

Proposition. $\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{k}\binom{2r-k}{k}/4^k = \binom{2r}{2k}\binom{2k}{k}/4^k$ *for integer* k.

Proof. Let k be an arbitrary integer. We must show that

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{k}\binom{2r-k}{k}\bigg/4^k = \binom{2r}{2k}\binom{2k}{k}\bigg/4^k.$$

In the case that $k<0$, we have

$$\begin{aligned}\binom{r}{k}\binom{r-1/2}{k} &= 0\\ &= \binom{2r}{k}\binom{2r-k}{k}\bigg/4^k;\end{aligned}$$

and in the case that $k=0$, we have

$$\begin{aligned}\binom{r}{k}\binom{r-1/2}{k} &= 1\\ &= \binom{2r}{k}\binom{2r-k}{k}\bigg/4^k.\end{aligned}$$

That is, in the case that $k\le 0$, we have

$$\begin{aligned}\binom{r}{k}\binom{r-1/2}{k} &= \delta_{k0}\\ &= \binom{2r}{k}\binom{2r-k}{k}\bigg/4^k.\end{aligned}$$

In the case that $k = r$, we have

$$\begin{aligned}
\binom{r}{k}\binom{r-1/2}{k} &= \binom{k-1/2}{k} \\
&= \frac{1}{(k-1/2+1)B(k+1,k-1/2-k+1)} \\
&= \frac{1}{(k+1/2)B(k+1,1/2)} \\
&= \frac{\Gamma(k+1+1/2)}{(k+1/2)\Gamma(k+1)\Gamma(1/2)} \\
&= \frac{\Gamma(k+1/2)}{\Gamma(k+1)\Gamma(1/2)} \\
&= \frac{\Gamma(1/2)\Gamma(k+1/2)}{\Gamma(1/2)^2\Gamma(k+1)} \\
&= \frac{\Gamma(1/2)\Gamma(k+1/2)}{\pi\Gamma(k+1)} \\
&= \frac{2(k+1)\Gamma(3/2)\Gamma(k+1/2)}{\pi\Gamma(3/2+k+1/2)} \\
&= \frac{2(k+1)B(3/2,k+1/2)}{\pi} \\
&= 2\Big/\frac{1}{(k+1)B(1/2+1,k-1/2+1)}\pi \\
&= 2\Big/\binom{k}{1/2}\pi \\
&= 2^{2k+1}\Big/\binom{k}{1/2}\pi 4^k \\
&= \binom{2k}{k}\Big/4^k.
\end{aligned}$$

It remains to consider the case when $0 < k \neq r$. Assuming

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{k}\binom{2r-k}{k}\Big/4^k$$

we must show that

$$\binom{r}{k+1}\binom{r-1/2}{k+1} = \binom{2r}{k+1}\binom{2r-(k+1)}{k+1}\Big/4^{k+1}.$$

As a corollary, note that from Eqs. (7) and (8) with $0 \neq k \neq r$ we have

$$\begin{aligned}
\binom{r}{k} &= \frac{r}{k}\binom{r-1}{k-1} \\
&= \frac{r}{k}\frac{r-k+1}{r}\binom{r}{k-1} \\
&= \frac{r-k+1}{k}\binom{r}{k-1}.
\end{aligned}$$

Then

$$
\begin{aligned}
&\binom{r}{k+1}\binom{r-1/2}{k+1} \\
&\quad = \binom{r}{k+1}\binom{r-1/2}{k+1} \\
&\quad = \frac{r-(k+1)+1}{k+1}\binom{r}{(k+1)-1}\frac{r-1/2-(k+1)+1}{k+1}\binom{r-1/2}{(k+1)-1} \\
&\quad = \frac{(r-k)(r-k-1/2)}{(k+1)^2}\binom{r}{k}\binom{r-1/2}{k} \\
&\quad = \frac{(r-k)(r-k-1/2)}{(k+1)^2}\binom{2r}{k}\binom{2r-k}{k}\Big/4^k \\
&\quad = \frac{4(r-k)(r-k-1/2)}{4(k+1)^2}\binom{2r}{k}\binom{2r-k}{k}\Big/4^k \\
&\quad = \frac{2(2r-k)(r-k-1/2)}{4(k+1)^2}\binom{2r}{k}\frac{2(r-k)}{2r-k}\binom{2r-k}{k}\Big/4^k \\
&\quad = \frac{2(2r-k)(r-k-1/2)}{4(k+1)^2}\binom{2r}{k}\binom{2r-k-1}{k}\Big/4^k \\
&\quad = \frac{1}{4}\frac{2r-(k+1)+1}{k+1}\binom{2r}{(k+1)-1}\frac{(2r-k-1)-(k+1)+1}{k+1}\binom{2r-k-1}{(k+1)-1}\Big/4^k \\
&\quad = \frac{1}{4}\binom{2r}{k+1}\binom{2r-k-1}{k+1}\Big/4^k \\
&\quad = \binom{2r}{k+1}\binom{2r-k-1}{k+1}\Big/4^{k+1} \\
&\quad = \binom{2r}{k+1}\binom{2r-(k+1)}{k+1}\Big/4^{k+1}.
\end{aligned}
$$

That is, for k an arbitrary integer

$$\binom{r}{k}\binom{r-1/2}{k} = \binom{2r}{k}\binom{2r-k}{k}\Big/4^k.$$

And finally, from Eq. (20)

$$
\begin{aligned}
&\binom{2r}{k}\binom{2r-k}{k}\Big/4^k \\
&\quad = \binom{2r}{k}\binom{2r-k}{2k-k}\Big/4^k \\
&\quad = \binom{2r}{2k}\binom{2k}{k}\Big/4^k
\end{aligned}
$$

as we needed to show. □

For the special case $r = -1/2$, we have the simpler equalities

$$\begin{aligned}\binom{-1/2}{k}\binom{-1/2-1/2}{k} &= \binom{-1/2}{k}\binom{-1}{k}\\ &= \binom{-2/2}{k}\binom{-2/2-k}{k}\Big/4^k\\ &= \binom{-1}{k}\binom{-1-k}{k}\Big/4^k\\ &= \binom{-2/2}{2k}\binom{2k}{k}\Big/4^k\\ &= \binom{-1}{2k}\binom{2k}{k}\Big/4^k\end{aligned}$$

if and only if, from Eq. (17)

$$\begin{aligned}\binom{-1/2}{k} &= \frac{\binom{-1}{2k}\binom{2k}{k}}{4^k\binom{-1}{k}}\\ &= \frac{(-1)^{2k}\binom{2k-(-1)-1}{2k}\binom{2k}{k}}{4^k\binom{-1}{k}}\\ &= \frac{\binom{2k}{k}}{4^k\binom{-1}{k}}\\ &= \frac{\binom{2k}{k}}{4^k(-1)^k\binom{k-(-1)-1}{k}}\\ &= \frac{\binom{2k}{k}}{4^k(-1)^k}\\ &= \left(\frac{-1}{4}\right)^k\binom{2k}{k}.\end{aligned}$$

That is, we have the simpler formula

$$\binom{-1/2}{k} = \left(\frac{-1}{4}\right)^k\binom{2k}{k}.$$

▶ **48.** [*M25*] Show that

$$\sum_{k\geq 0}\binom{n}{k}\frac{(-1)^k}{k+x} = \frac{n!}{x(x+1)\ldots(x+n)} = \frac{1}{x\binom{n+x}{n}},$$

if the denominators are not zero. [Note that this formula gives us the reciprocal of a binomial coefficient, as well as the partial fraction expansion of $1/x(x+1)\ldots(x+n)$.]

Proposition. $\sum_{k\geq 0}\binom{n}{k}\frac{(-1)^k}{k+x} = \frac{n!}{\prod_{0\leq j\leq n} x+j} = \frac{1}{x\binom{n+x}{n}}$.

Proof. Let n and k be arbitrary, nonnegative integers. We must show that

$$\sum_{k\geq 0}\binom{n}{k}\frac{(-1)^k}{k+x} = \frac{n!}{\prod_{0\leq j\leq n} x+j} = \frac{1}{x\binom{n+x}{n}}.$$

First, note that

$$\begin{aligned}
\frac{n!}{\prod_{0\le j\le n} x+j} &= \frac{n!}{x^{\overline{n+1}}} \\
&= \frac{n!}{(x+n+1-1)^{\underline{n+1}}} \\
&= \frac{n!}{(x+n)^{\underline{n+1}}} \\
&= \frac{n!(x+n-(n+1))!}{(x+n)!} \\
&= \frac{n!(x-1)!}{(x+n)!} \\
&= \frac{n!x!}{x(n+x)!} \\
&= \frac{n!(n+x-n)!}{x(n+x)!} \\
&= \frac{1}{x\binom{n+x}{n}}.
\end{aligned}$$

Then, if $n = 0$,

$$\begin{aligned}
\sum_{k\ge 0} \binom{n}{k} \frac{(-1)^k}{k+x} &= \binom{0}{0} \frac{(-1)^0}{0+x} \\
&= \frac{1}{x} \\
&= \frac{1}{x\binom{0+x}{0}}.
\end{aligned}$$

Assuming

$$\sum_{k\ge 0} \binom{n}{k} \frac{(-1)^k}{k+x} = \frac{n!}{\prod_{0\le j\le n} x+j} = \frac{1}{x\binom{n+x}{n}}$$

we must show that

$$\sum_{k\ge 0} \binom{n+1}{k} \frac{(-1)^k}{k+x} = \frac{(n+1)!}{\prod_{0\le j\le n+1} x+j} = \frac{1}{x\binom{n+1+x}{n+1}}.$$

But

$$
\begin{aligned}
&\sum_{k\ge 0}\binom{n+1}{k}\frac{(-1)^k}{x+k}\\
&\quad=\sum_{0\le k\le n+1}\binom{n+1}{k}\frac{(-1)^k}{x+k}\\
&\quad=\binom{n+1}{n+1}\frac{(-1)^{n+1}}{x+n+1}+\sum_{0\le k\le n}\binom{n+1}{k}\frac{(-1)^k}{x+k}\\
&\quad=\frac{(-1)^{n+1}}{x+n+1}+\sum_{0\le k\le n}\binom{n+1}{k}\frac{(-1)^k}{x+k}\\
&\quad=\frac{(-1)^{n+1}}{x+n+1}+\sum_{0\le k\le n}\left(\binom{n}{k}+\binom{n}{k-1}\right)\frac{(-1)^k}{x+k}\\
&\quad=\frac{(-1)^{n+1}}{x+n+1}+\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{x+k}+\sum_{0\le k\le n}\binom{n}{k-1}\frac{(-1)^k}{x+k}\\
&\quad=\frac{(-1)^{n+1}}{x+n+1}+\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{x+k}+\sum_{-1\le k\le n-1}\binom{n}{k}\frac{(-1)^{k+1}}{x+k+1}\\
&\quad=\frac{(-1)^{n+1}}{x+n+1}+\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{x+k}-\sum_{-1\le k\le n-1}\binom{n}{k}\frac{(-1)^k}{(x+1)+k}\\
&\quad=\frac{(-1)^{n+1}}{x+n+1}-\binom{n}{-1}\frac{-1}{x}+\binom{n}{n}\frac{(-1)^n}{x+n+1}+\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{x+k}-\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{(x+1)+k}\\
&\quad=\frac{(-1)^{n+1}+(-1)^n}{x+n+1}-\binom{n}{-1}\frac{-1}{x}+\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{x+k}-\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{(x+1)+k}\\
&\quad=\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{x+k}-\sum_{0\le k\le n}\binom{n}{k}\frac{(-1)^k}{(x+1)+k}\\
&\quad=\frac{1}{x\binom{n+x}{n}}-\frac{1}{(x+1)\binom{n+(x+1)}{n}}\\
&\quad=\frac{n!x!}{x(n+x)!}-\frac{n!(x+1)!}{(x+1)(n+x+1)!}\\
&\quad=\frac{n!x!(x+1)(n+x+1)}{x(x+1)(n+x)!(n+x+1)}-\frac{n!x(x+1)!}{x(x+1)(n+x+1)!}\\
&\quad=\frac{n!(x-1)!(n+x+1)}{(n+x+1)!}-\frac{n!x!}{(n+x+1)!}\\
&\quad=\frac{n!(x-1)!(n+x+1)-n!x!}{(n+x+1)!}\\
&\quad=\frac{n!(x-1)!(n+x+1)-n!(x-1)!x}{(n+x+1)!}\\
&\quad=\frac{n!(x-1)!(n+1)}{(n+x+1)!}\\
&\quad=\frac{(n+1)!(x-1)!}{(n+x+1)!}\\
&\quad=\frac{(n+1)!((n+1+x)-(n+1))!}{x(n+1+x)!}\\
&\quad=\frac{1}{x\binom{n+1+x}{n+1}}
\end{aligned}
$$

as we needed to show. □

49. [*M20*] Show that the identity $(1+x)^r = (1-x^2)^r(1-x)^{-r}$ implies a relation on binomial coefficients.

Given

$$(1+x)^r = (1-x^2)^r(1-x)^{-r}$$

and the binomial theorem, we have that

$$\begin{aligned}
\sum_{0\le m}\binom{r}{m}x^m &= (1+x)^r \\
&= (1-x^2)^r(1-x)^{-r} \\
&= \left(\sum_{0\le k\le r}\binom{r}{k}(-1)^k x^{2k}\right)\left(\sum_{0\le l}\binom{-r}{l}(-1)^l x^l\right) \\
&= \sum_{0\le k\le r}\sum_{0\le l}\binom{r}{k}\binom{-r}{l}(-1)^{k+l}x^{2k+l} \\
&= \sum_{0\le k\le r}\sum_{0\le m-2k}\binom{r}{k}\binom{-r}{m-2k}(-1)^{k+m-2k}x^m \\
&= \sum_{0\le k\le r}\sum_{2k\le m}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m-k}x^m \\
&= \sum_{0\le k\le r}\sum_{2k\le m}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m+k}x^m \\
&= \sum_{0\le k\le r}\left(\left(\sum_{2k\le m}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m+k}x^m\right)+0\right) \\
&= \sum_{0\le k\le r}\left(\left(\sum_{2k\le m}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m+k}x^m\right)+\left(\sum_{0\le m<2k}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m+k}x^m\right)\right) \\
&= \sum_{0\le k\le r}\sum_{0\le m}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m+k}x^m \\
&= \sum_{0\le m}\sum_{0\le k\le r}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m+k}x^m
\end{aligned}$$

which implies the relation

$$\binom{r}{m} = \sum_{0\le k\le r}\binom{r}{k}\binom{-r}{m-2k}(-1)^{m+k}$$

for integer m.

50. [*M20*] Prove Abel's formula, Eq. (16), in the special case $x+y=0$.

Proposition. *$(x+y)^n = \sum_{0\le k\le n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k}$ for integer $n\ge 0$, $x\ne 0$, $x+y=0$.*

Proof. Let n be an arbitrary nonnegative integer and x, y, z arbitrary reals such that $x\ne 0$, $x+y=0$. We must show that

$$(x+y)^n = \sum_{0\le k\le n}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k};$$

or equivalently, since $x + y = 0$, that

$$\delta_{n,0} = \sum_{0 \le k \le n} \binom{n}{k} x(x - kz)^{k-1}(-x + kz)^{n-k}.$$

But by the binomial theorem and Eq. (34)

$$\begin{aligned}
&\sum_{0 \le k \le n} \binom{n}{k} x(x - kz)^{k-1}(-x + kz)^{n-k} \\
&\quad = \sum_{0 \le k \le n} \binom{n}{k} x(x - kz)^{k-1}(-1)^{n-k}(x - kz)^{n-k} \\
&\quad = \sum_{0 \le k \le n} \binom{n}{k} (-1)^{n-k}(x - kz)^{k-1+n-k} x \\
&\quad = \sum_{0 \le k \le n} \binom{n}{k} (-1)^{n-k}(x - kz)^{n-1} x \\
&\quad = \sum_{0 \le k \le n} \binom{n}{k} (-1)^{n-k} x(x - zk)^{n-1} \\
&\quad = \sum_{0 \le k \le n} \binom{n}{k} (-1)^{n-k} x \sum_{0 \le m \le n-1} \binom{n-1}{m} x^{n-1-m}(-zk)^m \\
&\quad = \sum_{0 \le k \le n} \binom{n}{k} (-1)^{n-k} \sum_{0 \le m \le n-1} \binom{n-1}{m} x^{n-m}(-1)^m z^m k^m \\
&\quad = n! \binom{n-1}{n} x^{n-n}(-1)^n z^n \\
&\quad = \delta_{n,0}
\end{aligned}$$

as we needed to show. □

51. [*M21*] Prove Abel's formula, Eq. (16), by writing $y = (x + y) - x$, expanding the right-hand side in powers of $(x + y)$, and applying the result of the previous exercise.

Proposition. $(x + y)^n = \sum_{0 \le k} \binom{n}{k} x(x - kz)^{k-1}(y + kz)^{n-k}$ *for integer* $n \ge 0$, $x \ne 0$, $x + y = 0$.

Proof. Let n be an arbitrary nonnegative integer and x, y, z arbitrary reals such that $x \ne 0$, $x + y = 0$. We must show that

$$\sum_{0 \le k} \binom{n}{k} x(x - kz)^{k-1}(y + kz)^{n-k} = (x + y)^n;$$

or equivalently, since $x + y = x + y \Longrightarrow y = (x + y) - x$, that

$$\sum_{0 \le k} \binom{n}{k} x(x - kz)^{k-1}((x + y) - x + kz)^{n-k} = (x + y)^n.$$

But from Eq. (6), the binomial theorem, Eq. (20), and exercise 50

$$\begin{aligned}
&\sum_{0\le k}\binom{n}{k}x(x-kz)^{k-1}((x+y)-x+kz)^{n-k}\\
&\quad=\sum_{0\le k}\binom{n}{n-k}x(x-kz)^{k-1}((x+y)-x+kz)^{n-k}\\
&\quad=\sum_{0\le k}\binom{n}{n-k}x(x-kz)^{k-1}((x+y)+(-x+kz))^{n-k}\\
&\quad=\sum_{0\le k}\binom{n}{n-k}x(x-kz)^{k-1}\sum_{0\le m}\binom{n-k}{m}(x+y)^m(-x+kz)^{n-k-m}\\
&\quad=\sum_{0\le k}\sum_{0\le m}\binom{n}{n-k}\binom{n-k}{m}(x+y)^m x(x-kz)^{k-1}(-x+kz)^{n-k-m}\\
&\quad=\sum_{0\le k}\sum_{0\le m}\binom{n}{m}\binom{n-m}{n-m-k}(x+y)^m x(x-kz)^{k-1}(-x+kz)^{n-k-m}\\
&\quad=\sum_{0\le m}\binom{n}{m}(x+y)^m\sum_{0\le k}\binom{n-m}{n-m-k}x(x-kz)^{k-1}(-x+kz)^{n-k-m}\\
&\quad=\sum_{0\le m}\binom{n}{m}(x+y)^m\delta_{n-m,0}\\
&\quad=\binom{n}{n}(x+y)^n\\
&\quad=(x+y)^n
\end{aligned}$$

as we needed to show. □

A. Hurwitz, *Acta Mathematica* **26** (1902), 199–203.

52. [*HM11*] Prove that Abel's binomial formula (16) is not always valid when n is not a nonnegative integer, by evaluating the right-hand side when $n = x = -1$, $y = z = 1$.

We may prove that Abel's binomial formula

$$(x+y)^n=\sum_{0\le k}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k}$$

is not always valid when $n < 0$ by evaluating the particular case when $n = x = -1$, $y = z = 1$;

and

$$\begin{aligned}
&\sum_{0\le k}\binom{n}{k}x(x-kz)^{k-1}(y+kz)^{n-k}\\
&\quad=\sum_{0\le k}\binom{-1}{k}(-1)((-1)-k)^{k-1}(1+k)^{-1-k}\\
&\quad=\sum_{0\le k}\binom{-1}{k}(-1)^k(k+1)^{k-1}(k+1)^{-k-1}\\
&\quad=\sum_{0\le k}\binom{-1}{k}(-1)^k(k+1)^{-2}\\
&\quad=\sum_{0\le k}(k+1)^{-2} && \text{from Eq. (17)}\\
&\quad=\sum_{1\le k}k^{-2}
\end{aligned}$$

where $\sum_{1\le k}k^{-2}$ is the Riemann zeta function $\zeta(-2)=\pi^2/6\neq 0=(-1+1)^{-1}=(x+y)^n$.

53. [*M25*] (a) Prove the following identity by induction on m, where m and n are integers:

$$\sum_{k=0}^{m}\binom{r}{k}\binom{s}{n-k}(nr-(r+s)k)=(m+1)(n-m)\binom{r}{m+1}\binom{s}{n-m}.$$

(b) Making use of important relations from exercise 47,

$$\binom{-1/2}{n}=\frac{(-1)^n}{2^{2n}}\binom{2n}{n},\qquad \binom{1/2}{n}=\frac{(-1)^{n-1}}{2^{2n}(2n-1)}\binom{2n}{n}=\frac{(-1)^{n-1}}{2^{2n-1}(2n-1)}\binom{2n-1}{n}-\delta_{n0},$$

show that the following formula can be obtained as a special case of the identity in part (a):

$$\sum_{k=0}^{m}\binom{2k-1}{k}\binom{2n-2k}{n-k}\frac{-1}{2k-1}=\frac{n-m}{2n}\binom{2m}{m}\binom{2n-2m}{n-m}+\frac{1}{2}\binom{2n}{n}.$$

(This result is considerably more general than Eq. (26) in the case $r=-1$, $s=0$, $t=-2$.)

a) We may prove the identity by induction.

Proposition. $\sum_{k=0}^{m}\binom{r}{k}\binom{s}{n-k}(nr-(r+s)k)=(m+1)(n-m)\binom{r}{m+1}\binom{s}{n-m}$ *for integers m, n.*

Proof. Let m and n be arbitrary integers such that m is nonnegative. We must show that

$$\sum_{k=0}^{m}\binom{r}{k}\binom{s}{n-k}(nr-(r+s)k)=(m+1)(n-m)\binom{r}{m+1}\binom{s}{n-m}.$$

If $m = 0$

$$\begin{aligned}
\sum_{k=0}^{m} \binom{r}{k}\binom{s}{n-k}(nr-(r+s)k) &= \sum_{k=0}^{0} \binom{r}{k}\binom{s}{n-k}(nr-(r+s)k) \\
&= \binom{r}{0}\binom{s}{n-0}(nr-(r+s)(0)) \\
&= \binom{r}{0}\binom{s}{n}(nr) \\
&= \binom{s}{n}(nr) \\
&= nr\binom{s}{n} \\
&= nr\frac{1!(r-1)!}{1!(r-1)!}\binom{s}{n} \\
&= nr\frac{(r-1)!}{1!(r-1)!}\binom{s}{n} \\
&= n\frac{r!}{1!(r-1)!}\binom{s}{n} \\
&= n\binom{r}{1}\binom{s}{n} \\
&= (1)(n)\binom{r}{1}\binom{s}{n} \\
&= (0+1)(n-0)\binom{r}{0+1}\binom{s}{n-0} \\
&= (m+1)(n-m)\binom{r}{m+1}\binom{s}{n-m}.
\end{aligned}$$

Then, assuming

$$\sum_{k=0}^{m} \binom{r}{k}\binom{s}{n-k}(nr-(r+s)k) = (m+1)(n-m)\binom{r}{m+1}\binom{s}{n-m}$$

we must show that

$$\begin{aligned}
&\sum_{k=0}^{m+1} \binom{r}{k}\binom{s}{n-k}(nr-(r+s)k) \\
&\quad = ((m+1)+1)(n-(m+1))\binom{r}{(m+1)+1}\binom{s}{n-(m+1)}.
\end{aligned}$$

But from the inductive hypothesis as well as Eqs. (7) and (8)

$$
\begin{aligned}
&\sum_{k=0}^{m+1} \binom{r}{k}\binom{s}{n-k}(nr-(r+s)k) \\
&\quad= \sum_{k=0}^{m} \binom{r}{k}\binom{s}{n-k}(nr-(r+s)k) + \binom{r}{m+1}\binom{s}{n-(m+1)}(nr-(r+s)(m+1)) \\
&\quad= (m+1)(n-m)\binom{r}{m+1}\binom{s}{n-m} + \binom{r}{m+1}\binom{s}{n-(m+1)}(nr-(r+s)(m+1)) \\
&\quad= (m+1)(n-m)\binom{r}{m+1}\frac{s}{n-m}\binom{s-1}{n-m-1} \\
&\qquad+ \binom{r}{m+1}\binom{s}{n-(m+1)}(nr-(r+s)(m+1)) \\
&\quad= (m+1)s\binom{r}{m+1}\binom{s-1}{n-m-1} + \binom{r}{m+1}\binom{s}{n-(m+1)}(nr-(r+s)(m+1)) \\
&\quad= (m+1)s\binom{r}{m+1}\frac{s-(n-m-1)}{s}\binom{s}{n-m-1} \\
&\qquad+ \binom{r}{m+1}\binom{s}{n-(m+1)}(nr-(r+s)(m+1)) \\
&\quad= (m+1)(s-n+m+1)\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\qquad+ \binom{r}{m+1}\binom{s}{n-(m+1)}(nr-(r+s)(m+1)) \\
&\quad= ((m+1)(s-n+m+1)+(nr-(r+s)(m+1)))\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\quad= ((m+1)((s-n+m+1)-(r+s))+nr)\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\quad= ((m+1)(-r-n+m+1)+nr)\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\quad= (nr+(-r-n+m+1)(m+1))\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\quad= (nr-r(m+1)-n(m+1)+(m+1)^2)\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\quad= (n-(m+1))(r-(m+1))\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\quad= r(n-(m+1))\frac{r-(m+1)}{r}\binom{r}{m+1}\binom{s}{n-(m+1)} \\
&\quad= r(n-(m+1))\binom{r-1}{m+1}\binom{s}{n-(m+1)} \\
&\quad= ((m+1)+1)(n-(m+1))\frac{r}{(m+1)+1}\binom{r-1}{m+1}\binom{s}{n-(m+1)} \\
&\quad= ((m+1)+1)(n-(m+1))\binom{r}{(m+1)+1}\binom{s}{n-(m+1)}
\end{aligned}
$$

as we needed to show. □

b) We may derive the formula as a special case of the identity in part (a).

Given the relations from exercise 47, since

$$\begin{aligned}\binom{1/2}{n} &= \frac{(-1)^{n-1}}{2^{2n-1}(2n-1)}\binom{2n-1}{n} - \delta_{n0} \\ \iff \quad \binom{2n-1}{n} &= \left(\binom{1/2}{n} + \delta_{n0}\right)\frac{2^{2n-1}(2n-1)}{(-1)^{n-1}}\end{aligned}$$

and

$$\begin{aligned}\binom{-1/2}{n} &= \frac{(-1)^n}{2^{2n}}\binom{2n}{n} \\ \iff \quad \binom{2n}{n} &= \binom{-1/2}{n}\frac{2^{2n}}{(-1)^n}\end{aligned}$$

we have that

$$\begin{aligned}&\sum_{k=0}^{m}\binom{2k-1}{k}\binom{2n-2k}{n-k}\frac{-1}{2k-1} \\ &\quad= \sum_{k=0}^{m}\left(\binom{1/2}{k} + \delta_{k0}\right)\frac{2^{2k-1}(2k-1)}{(-1)^{k-1}}\binom{2n-2k}{n-k}\frac{-1}{2k-1} \\ &\quad= \sum_{k=0}^{m}\left(\binom{1/2}{k} + \delta_{k0}\right)2^{2k-1}(-1)^k\binom{2n-2k}{n-k} \\ &\quad= \sum_{k=0}^{m}\left(\binom{1/2}{k} + \delta_{k0}\right)2^{2k-1}(-1)^k\binom{2(n-k)}{n-k} \\ &\quad= \sum_{k=0}^{m}\left(\binom{1/2}{k} + \delta_{k0}\right)2^{2k-1}(-1)^k\binom{-1/2}{n-k}\frac{2^{2(n-k)}}{(-1)^{n-k}} \\ &\quad= \sum_{k=0}^{m}\left(\binom{1/2}{k} + \delta_{k0}\right)2^{2n-1}(-1)^n\binom{-1/2}{n-k} \\ &\quad= (-1)^n 2^{2n-1}\sum_{k=0}^{m}\left(\binom{1/2}{k}\binom{-1/2}{n-k} + \delta_{k0}\binom{-1/2}{n-k}\right) \\ &\quad= (-1)^n 2^{2n-1}\left(\sum_{k=0}^{m}\binom{1/2}{k}\binom{-1/2}{n-k} + \binom{-1/2}{n}\right).\end{aligned}$$

Then, setting $r = \frac{1}{2}$, $s = -\frac{1}{2}$ in the result of (a) yields

$$\begin{aligned}&\sum_{k=0}^{m}\binom{1/2}{k}\binom{-1/2}{n-k}(n/2 - (1/2 - 1/2)k) \\ &\quad= \sum_{k=0}^{m}\binom{1/2}{k}\binom{-1/2}{n-k}(n/2) \\ &\quad= (m+1)(n-m)\binom{1/2}{m+1}\binom{-1/2}{n-m}\end{aligned}$$

so that, again with the relations from exercise 47, as well as Eqs. (7) and (8),

$$
\begin{aligned}
&(-1)^n 2^{2n-1}\left(\sum_{k=0}^{m}\binom{1/2}{k}\binom{-1/2}{n-k}+\binom{-1/2}{n}\right)\\
&\quad= (-1)^n 2^{2n-1}\left(\frac{2(m+1)(n-m)}{n}\binom{1/2}{m+1}\binom{-1/2}{n-m}+\binom{-1/2}{n}\right)\\
&\quad= (-1)^n 2^{2n-1}\frac{2(m+1)(n-m)}{n}\frac{(-1)^{m+1-1}}{2^{2(m+1)}(2(m+1)-1)}\binom{2(m+1)}{m+1}\frac{(-1)^{n-m}}{2^{2(n-m)}}\binom{2(n-m)}{n-m}\\
&\qquad+(-1)^n 2^{2n-1}\frac{(-1)^n}{2^{2n}}\binom{2n}{n}\\
&\quad= \frac{(m+1)(n-m)}{2^2 n(2m+1)}\binom{2m+2}{m+1}\binom{2n-2m}{n-m}+\frac{1}{2}\binom{2n}{n}\\
&\quad= \frac{(m+1)(n-m)}{2^2 n(2m+1)}\frac{2m+2}{m+1}\binom{2m+1}{m}\binom{2n-2m}{n-m}+\frac{1}{2}\binom{2n}{n}\\
&\quad= \frac{(m+1)(n-m)}{2n(2m+1)}\binom{2m+1}{m}\binom{2n-2m}{n-m}+\frac{1}{2}\binom{2n}{n}\\
&\quad= \frac{(m+1)(n-m)}{2n(2m+1)}\frac{2m+1}{2m+1-m}\binom{2m+1-1}{m}\binom{2n-2m}{n-m}+\frac{1}{2}\binom{2n}{n}\\
&\quad= \frac{n-m}{2n}\binom{2m}{m}\binom{2n-2m}{n-m}+\frac{1}{2}\binom{2n}{n}.
\end{aligned}
$$

Hence

$$\sum_{k=0}^{m}\binom{2k-1}{k}\binom{2n-2k}{n-k}\frac{-1}{2k-1}=\frac{n-m}{2n}\binom{2m}{m}\binom{2n-2m}{n-m}+\frac{1}{2}\binom{2n}{n}.$$

54. [*M21*] Consider Pascal's triangle (as shown in Table 1) as a matrix. What is the *inverse* of that matrix?

Let

$$A_{n+1}=[a_{ij}]_{n+1}=\left[\binom{i}{j}\right]_{n+1}=\begin{bmatrix}\binom{0}{0} & \binom{0}{1} & \cdots & \binom{0}{n}\\ \binom{1}{0} & \binom{1}{1} & \cdots & \binom{1}{n}\\ \vdots & \vdots & \ddots & \vdots\\ \binom{n}{0} & \binom{n}{1} & \cdots & \binom{n}{n}\end{bmatrix}_{n+1}$$

represent Pascal's triangle as a matrix. We want to find another matrix $B_{n+1}=[b_{ij}]_{n+1}$ such that

$$A_{n+1}B_{n+1}=B_{n+1}A_{n+1}=I_{n+1}=[\delta_{ij}]_{n+1}\,.$$

But by the definition of matrix multiplication

$$\begin{aligned}
B_{n+1}A_{n+1} &= \sum_{0\le k\le n} b_{ik}a_{kj} \\
&= \sum_{0\le k\le n} b_{ik}\binom{k}{j} \\
&= \delta_{ij} \\
&= \sum_{0\le k\le i} \binom{i}{k}\binom{k}{j}(-1)^{i-k} && \text{from Eq. (33)} \\
&= \sum_{0\le k\le n} \binom{i}{k}\binom{k}{j}(-1)^{i+k} \\
&= \sum_{0\le k\le n} (-1)^{i+k}\binom{i}{k}\binom{k}{j}.
\end{aligned}$$

Hence, for arbitrary k, $0 \le k \le n$

$$b_{ik}\binom{k}{j} = (-1)^{i+k}\binom{i}{k}\binom{k}{j} \quad\Longleftrightarrow\quad b_{ik} = (-1)^{i+k}\binom{i}{k}$$

and in particular for $k = j$

$$b_{ij} = (-1)^{i+j}\binom{i}{j}$$

as we wanted to find, so that the inverse matrix is given by

$$B_{n+1} = [b_{ij}]_{n+1} = \left[(-1)^{i+j}\binom{i}{j}\right]_{n+1} = \begin{bmatrix} \binom{0}{0} & -\binom{0}{1} & \cdots & (-1)^n\binom{0}{n} \\ -\binom{1}{0} & \binom{1}{1} & \cdots & (-1)^{1+n}\binom{1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n\binom{n}{0} & (-1)^{n+1}\binom{n}{1} & \cdots & \binom{n}{n} \end{bmatrix}_{n+1}.$$

55. [*M21*] Considering each of Stirling's triangles (Table 2) as matrices, determine their inverses.

Let

$$A_{n+1} = [a_{ij}]_{n+1} = \left[\left[{i \atop j}\right]\right]_{n+1} = \begin{bmatrix} \left[{0 \atop 0}\right] & \left[{0 \atop 1}\right] & \cdots & \left[{0 \atop n}\right] \\ \left[{1 \atop 0}\right] & \left[{1 \atop 1}\right] & \cdots & \left[{1 \atop n}\right] \\ \vdots & \vdots & \ddots & \vdots \\ \left[{n \atop 0}\right] & \left[{n \atop 1}\right] & \cdots & \left[{n \atop n}\right] \end{bmatrix}_{n+1}$$

represent Stirling's triangle of the first kind as a matrix. We want to find another matrix $B_{n+1} = [b_{ij}]_{n+1}$ such that

$$A_{n+1}B_{n+1} = B_{n+1}A_{n+1} = I_{n+1} = [\delta_{ij}]_{n+1}.$$

But by the definition of matrix multiplication

$$\begin{aligned}
B_{n+1}A_{n+1} &= \sum_{0\le k\le n} b_{ik}a_{kj} \\
&= \sum_{0\le k\le n} b_{ik}\left[{k \atop j}\right] \\
&= \delta_{ij} \\
&= \delta_{ji} \\
&= \sum_{0\le k\le i} \left\{{i \atop k}\right\}\left[{k \atop j}\right](-1)^{i-k} && \text{from Eq. (47)} \\
&= \sum_{0\le k\le n} (-1)^{i+k}\left\{{i \atop k}\right\}\left[{k \atop j}\right].
\end{aligned}$$

Hence, for arbitrary k, $0 \le k \le n$

$$b_{ik}\left[{k \atop j}\right] = (-1)^{i+k}\left\{{i \atop k}\right\}\left[{k \atop j}\right] \quad\iff\quad b_{ik} = (-1)^{i+k}\left\{{i \atop k}\right\}$$

and in particular for $k = j$

$$b_{ij} = (-1)^{i+j}\left\{{i \atop j}\right\}$$

as we wanted to find, so that the inverse matrix is given by

$$B_{n+1} = [b_{ij}]_{n+1} = \left[(-1)^{i+j}\left\{{i \atop j}\right\}\right]_{n+1} = \begin{bmatrix} \left\{{0 \atop 0}\right\} & -\left\{{0 \atop 1}\right\} & \cdots & (-1)^n\left\{{0 \atop n}\right\} \\ -\left\{{1 \atop 0}\right\} & \left\{{1 \atop 1}\right\} & \cdots & (-1)^{1+n}\left\{{1 \atop n}\right\} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n\left\{{n \atop 0}\right\} & (-1)^{n+1}\left\{{n \atop 1}\right\} & \cdots & \left\{{n \atop n}\right\} \end{bmatrix}_{n+1}.$$

Similarly, let

$$A'_{n+1} = [a'_{ij}]_{n+1} = \left[\left\{{i \atop j}\right\}\right]_{n+1} = \begin{bmatrix} \left\{{0 \atop 0}\right\} & \left\{{0 \atop 1}\right\} & \cdots & \left\{{0 \atop n}\right\} \\ \left\{{1 \atop 0}\right\} & \left\{{1 \atop 1}\right\} & \cdots & \left\{{1 \atop n}\right\} \\ \vdots & \vdots & \ddots & \vdots \\ \left\{{n \atop 0}\right\} & \left\{{n \atop 1}\right\} & \cdots & \left\{{n \atop n}\right\} \end{bmatrix}_{n+1}$$

represent Stirling's triangle of the second kind as a matrix. We want to find another matrix $B'_{n+1} = [b'_{ij}]_{n+1}$ such that

$$A'_{n+1}B'_{n+1} = B'_{n+1}A'_{n+1} = I_{n+1} = [\delta_{ij}]_{n+1}.$$

But by the definition of matrix multiplication

$$\begin{aligned}
B'_{n+1}A'_{n+1} &= \sum_{0\le k\le n} b'_{ik}a'_{kj} \\
&= \sum_{0\le k\le n} b'_{ik}\left\{{k \atop j}\right\} \\
&= \delta_{ij} \\
&= \delta_{ji} \\
&= \sum_{0\le k\le i} \left[{i \atop k}\right]\left\{{k \atop j}\right\}(-1)^{i-k} && \text{from Eq. (47)} \\
&= \sum_{0\le k\le n} (-1)^{i+k}\left[{i \atop k}\right]\left\{{k \atop j}\right\}.
\end{aligned}$$

Hence, for arbitrary k, $0 \le k \le n$

$$b'_{ik}\left\{{k \atop j}\right\} = (-1)^{i+k}\left[{i \atop k}\right]\left\{{k \atop j}\right\} \quad\Longleftrightarrow\quad b'_{ik} = (-1)^{i+k}\left[{i \atop k}\right]$$

and in particular for $k = j$

$$b'_{ij} = (-1)^{i+j}\left[{i \atop j}\right]$$

as we wanted to find, so that the inverse matrix is given by

$$B'_{n+1} = [b'_{ij}]_{n+1} = \left[(-1)^{i+j}\left[{i \atop j}\right]\right]_{n+1} = \begin{bmatrix} \left[{0 \atop 0}\right] & -\left[{0 \atop 1}\right] & \cdots & (-1)^n\left[{0 \atop n}\right] \\ -\left[{1 \atop 0}\right] & \left[{1 \atop 1}\right] & \cdots & (-1)^{1+n}\left[{1 \atop n}\right] \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^n\left[{n \atop 0}\right] & (-1)^{n+1}\left[{n \atop 1}\right] & \cdots & \left[{n \atop n}\right] \end{bmatrix}_{n+1}.$$

56. [*20*] (*The combinatorial number system.*) For each integer $n = 0, 1, 2, \ldots, 20$, find three integers a, b, c for which $n = \binom{a}{3} + \binom{b}{2} + \binom{c}{1}$ and $a > b > c \ge 0$. Can you see how this pattern can be continued for higher values of n?

We may evaluate the largest values for a, b, and c that satisfy the constraint such that $a > b > c \ge 0$.

a	b	c	n
2	1	0	0
3	1	0	1
3	2	0	2
3	2	1	3
4	1	0	4
4	2	0	5
4	2	1	6
4	3	0	7
4	3	1	8
4	3	2	9
5	1	0	10
5	2	0	11
5	2	1	12
5	3	0	13
5	3	1	14
5	3	2	15
5	4	0	16
5	4	1	17
5	4	2	18
5	4	3	19
6	1	0	20

This pattern can be continued for higher values of n by the following method: let a be the largest integer that satisfies $\binom{a}{3} \le n$; b the largest that satisfies $\binom{b}{2} \le n - \binom{a}{3}$; and c that satisfies $\binom{c}{1} \le n - \binom{a}{3} - \binom{b}{2}$.

▶ **57.** [*M22*] Show that the coefficient a_m in Stirling's attempt at generalizing the factorial function, Eq. 1.2.5-(12), is

$$\frac{(-1)^m}{m!}\sum_{k\ge 1}(-1)^k\binom{m-1}{k-1}\ln k.$$

Proposition. $a_m = \frac{(-1)^m}{m!}\sum_{1\le k}(-1)^k\binom{m-1}{k-1}\ln k$ *in Stirling's generalization of the factorial function.*

Proof. Stirling's generalization of the factorial function is

$$\ln n! = \sum_{0 \le m} a_{m+1} \prod_{0 \le j \le m} (n-j).$$

We must show that

$$a_m = \frac{(-1)^m}{m!} \sum_{1 \le k} (-1)^k \binom{m-1}{k-1} \ln k.$$

But given

$$\sum_{0 \le j} \binom{m}{j} (-1)^{m-j} = \sum_{0 \le j} \binom{m}{j} (-1)^{m-j}$$

we have

$$\begin{aligned}
\sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \ln n! &= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \sum_{0 \le k} a_{k+1} \prod_{0 \le j \le k} (n-j) \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \sum_{0 \le k} a_{k+1} j^{\underline{k+1}} && \text{from Eq. 1.2.5-18} \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \sum_{0 \le k} a_{k+1} \frac{j!}{(j-(k+1))!} && \text{from Eq. 1.2.5-21} \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \sum_{0 \le k} a_{k+1} (k+1)! \frac{j!}{(k+1)!(j-(k+1))!} \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \sum_{0 \le k} a_{k+1} (k+1)! \binom{j}{k+1} \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \left(\sum_{0 \le k} a_k k! \binom{j}{k} - a_0 0! \binom{j}{0} \right) \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \left(\sum_{0 \le k} a_k k! \binom{j}{k} - a_0 \right) \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \left(\sum_{0 \le k} a_k k! \binom{j}{k} - \frac{(-1)^0}{0!} \sum_{1 \le k} (-1)^k \binom{-1}{k-1} \ln k \right) \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \left(\sum_{0 \le k} a_k k! \binom{j}{k} - 0 \right) \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \sum_{0 \le k} a_k k! \binom{j}{k} \\
&= \sum_{0 \le j} \binom{m}{j} (-1)^{m-j} \sum_{k} k! a_k \binom{j}{k} \\
&= \sum_{k} k! a_k \sum j \binom{j}{k} \binom{m}{j} (-1)^{m-j} \\
&= m! a_m && \text{from Eq. (33)}
\end{aligned}$$

And so

$$\begin{aligned}
a_m &= \frac{1}{m!}\sum_{0\le j}\binom{m}{j}(-1)^{m-j}\ln n! \\
&= \frac{1}{m!}\sum_{0\le j}\binom{m}{j}(-1)^{m+j}\ln n! \\
&= \frac{(-1)^m}{m!}\sum_{0\le j}\binom{m}{j}(-1)^{j}\ln n! \\
&= \frac{(-1)^m}{m!}\sum_{0\le j}\binom{m}{j}(-1)^{j}\sum_{1\le k\le n}\ln k \\
&= \frac{(-1)^m}{m!}\sum_{0\le j}\sum_{1\le k\le n}\binom{m}{j}(-1)^{j}\ln k \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}\sum_{k\le j}\binom{m}{j}(-1)^{j}\ln k \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}\ln k\sum_{k\le j}\binom{m}{j}(-1)^{j} \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}\ln k\left(\sum_{j\le m}\binom{m}{j}(-1)^{j}-\sum_{j\le k-1}\binom{m}{j}(-1)^{j}\right) \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}\ln k\left((-1+1)^m-\sum_{j\le k-1}\binom{m}{j}(-1)^{j}\right) && \text{from Eq. (13)} \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}\ln k\left(0-\sum_{j\le k-1}\binom{m}{j}(-1)^{j}\right) \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}\ln k(-1)\sum_{j\le k-1}\binom{m}{j}(-1)^{j} \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}\ln k(-1)(-1)^{k-1}\binom{m-1}{k-1} && \text{from Eq. (18)} \\
&= \frac{(-1)^m}{m!}\sum_{1\le k}(-1)^{k}\binom{m-1}{k-1}\ln k.
\end{aligned}$$

Therefore

$$a_m = \frac{(-1)^m}{m!}\sum_{1\le k}(-1)^{k}\binom{m-1}{k-1}\ln k$$

as we needed to show. □

58. [*M23*] (H. A. Rothe, 1811.) In the notation of Eq. (40), prove the "q-nomial theorem":

$$(1+x)(1+qx)\dots(1+q^{n-1}x) = \sum_k \binom{n}{k}_q q^{k(k-1)/2}x^k.$$

Also find q-nomial generalizations of the fundamental identities (17) and (21).

In order to prove the "q-nomial theorem", we define

$$\binom{r}{k}_q = \frac{\prod_{1\le j\le k}(1-q^{r-j+1})}{\prod_{1\le j\le k}(1-q^{j})} = \prod_{1\le j\le k}\frac{1-q^{r-j+1}}{1-q^{j}}$$

and assume q-nomial symmetry

$$\binom{n}{k}_q = \binom{n}{n-k}_q$$

as well as the two q-Pascal identities

$$\binom{n+1}{k}_q = \binom{n}{k}_q + \binom{n}{k-1}_q q^{n+1-k}$$

and

$$\binom{n+1}{k}_q = \binom{n}{k}_q q^k + \binom{n}{k-1}_q.$$

Proposition. $\prod_{0\le k\le n-1}(1+q^k x) = \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k$.

Proof. Let n and q be an arbitrary nonnegative integer and real number, respectively. We must show that

$$\prod_{0\le k\le n-1}(1+q^k x) = \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k$$

If $n = 0$

$$\begin{aligned}\prod_{0\le k\le n-1}(1+q^k x) &= \prod_{0\le k\le -1}(1+q^k x)\\ &= 1\\ &= \frac{1}{1}\\ &= \frac{\prod_{1\le j\le 0}(1-q^{0-j+1})}{\prod_{1\le j\le 0}(1-q^j)}\\ &= \binom{0}{0}_q\\ &= \binom{0}{0}_q q^{(0)(-1)/2}x^0\\ &= \sum_{0\le k\le 0}\binom{0}{k}_q q^{k(k-1)/2}x^k\\ &= \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k.\end{aligned}$$

Then, assuming

$$\prod_{0\le k\le n-1}(1+q^k x) = \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k$$

we must show that

$$\prod_{0\le k\le n}(1+q^k x) = \sum_{0\le k\le n+1}\binom{n+1}{k}_q q^{k(k-1)/2}x^k.$$

But

$$\begin{aligned}
\prod_{0\le k\le n}(1+q^kx) &= (1+q^nx)\prod_{0\le k\le n-1}(1+q^kx)\\
&= (1+q^nx)\sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k\\
&= \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k + q^nx\sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k\\
&= \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k + \sum_{0\le k\le n}\binom{n}{k}_q q^{n+k(k-1)/2}x^{k+1}\\
&= \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k + \sum_{1\le k\le n+1}\binom{n}{k-1}_q q^{n+(k-1)(k-2)/2}x^k\\
&= \sum_{0\le k\le n+1}\binom{n}{k}_q q^{k(k-1)/2}x^k + \sum_{0\le k\le n+1}\binom{n}{k-1}_q q^{n+(k-1)(k-2)/2}x^k\\
&= \sum_{0\le k\le n+1}\left(\binom{n}{k}_q + \binom{n}{k-1}_q q^{n+(k-1)(k-2)/2-k(k-1)/2}\right) q^{k(k-1)/2}x^k\\
&= \sum_{0\le k\le n+1}\left(\binom{n}{k}_q + \binom{n}{k-1}_q q^{n+1-k}\right) q^{k(k-1)/2}x^k\\
&= \sum_{0\le k\le n+1}\binom{n+1}{k}_q q^{k(k-1)/2}x^k.
\end{aligned}$$

Hence

$$\prod_{0\le k\le n-1}(1+q^kx) = \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k$$

as we needed to show. □

We may also find q-nomial generalizations of the fundamental identities (17) and (21).

For (17), we have

$$\begin{aligned}
\binom{r}{k}_q &= \prod_{1\le j\le k} \frac{1-q^{r-j+1}}{1-q^j} \\
&= \prod_{1\le j\le k} \frac{-1}{-1}\frac{1-q^{r-j+1}}{1-q^j} \\
&= (-1)^k \prod_{1\le j\le k} -\frac{1-q^{r-j+1}}{1-q^j} \\
&= (-1)^k \prod_{1\le j\le k} q^{r-j+1}\frac{1-q^{-r+j-1}}{1-q^j} \\
&= (-1)^k \left(\prod_{1\le j\le k} \frac{q^{r-j+1}}{1-q^j}\right)\left(\prod_{1\le j\le k} 1-q^{-r+j-1}\right) \\
&= (-1)^k \left(\prod_{1\le j\le k} \frac{q^{r-j+1}}{1-q^j}\right)\left(\prod_{1\le -j+k+1\le k} 1-q^{-r+j-1}\right) \\
&= (-1)^k \left(\prod_{1\le j\le k} \frac{q^{r-j+1}}{1-q^j}\right)\left(\prod_{1\le -j+k+1\le k} 1-q^{k-r-(-j+k+1)}\right) \\
&= (-1)^k \left(\prod_{1\le j\le k} \frac{q^{r-j+1}}{1-q^j}\right)\left(\prod_{1\le j\le k} 1-q^{k-r-j}\right) \\
&= (-1)^k \prod_{1\le j\le k} q^{r-j+1}\frac{1-q^{k-r-j}}{1-q^j} \\
&= (-1)^k \frac{\prod_{1\le j\le k} q^r}{\prod_{1\le j\le k} q^{j-1}} \prod_{1\le j\le k} \frac{1-q^{k-r-j}}{1-q^j} \\
&= (-1)^k \frac{q^{kr}}{\sqrt{(q^{1-1}q^{k-1})^k}} \prod_{1\le j\le k} \frac{1-q^{k-r-j}}{1-q^j} \\
&= (-1)^k \frac{q^{kr}}{q^{k(k-1)/2}} \prod_{1\le j\le k} \frac{1-q^{k-r-j}}{1-q^j} \\
&= (-1)^k q^{kr-k(k-1)/2} \prod_{1\le j\le k} \frac{1-q^{k-r-j}}{1-q^j} \\
&= (-1)^k q^{kr-k(k-1)/2} \prod_{1\le j\le k} \frac{1-q^{k-r-1-j+1}}{1-q^j} \\
&= (-1)^k q^{kr-k(k-1)/2} \binom{k-r-1}{k}_q \\
&= (-1)^k \binom{k-r-1}{k}_q q^{kr-k(k-1)/2}
\end{aligned}$$

so that the q-nomial generalizations of the fundamental identity (17) is

$$\binom{r}{k}_q = (-1)^k \binom{k-r-1}{k}_q q^{kr-k(k-1)/2}.$$

For (21), the q-nomial theorem

$$\prod_{0\le k\le n-1}(1+q^k x) = \sum_{0\le k\le n}\binom{n}{k}_q q^{k(k-1)/2}x^k$$

gives us

$$\begin{aligned}
&\sum_{0\le n\le r+s}\binom{r+s}{n}_q q^{n(n-1)/2}x^n \\
&\quad= \prod_{0\le n\le r+s-1}(1+q^n x) \\
&\quad= \prod_{0\le k\le r-1}(1+q^k x)\prod_{r\le k\le r+s-1}(1+q^k x) \\
&\quad= \prod_{0\le k\le r-1}(1+q^k x)\prod_{0\le k-r\le s-1}(1+q^k x) \\
&\quad= \prod_{0\le k\le r-1}(1+q^k x)\prod_{0\le k\le s-1}(1+q^{k+r} x) \\
&\quad= \prod_{0\le k\le r-1}(1+q^k x)\prod_{0\le k\le s-1}(1+q^k q^r x) \\
&\quad= \left(\sum_{0\le k\le r}\binom{r}{k}_q q^{k(k-1)/2}x^k\right)\left(\sum_{0\le k\le s}\binom{s}{k}_q q^{k(k-1)/2}(q^r x)^k\right) \\
&\quad= \left(\sum_{0\le k\le r}\binom{r}{k}_q q^{k(k-1)/2}x^k\right)\left(\sum_{0\le k\le s}\binom{s}{k}_q q^{k(k-1)/2}q^{rk}x^k\right) \\
&\quad= \left(\sum_{0\le k\le r}\binom{r}{k}_q q^{k(k-1)/2}x^k\right)\left(\sum_{0\le n-k\le s}\binom{s}{n-k}_q q^{(n-k)(n-k-1)/2}q^{r(n-k)}x^{n-k}\right) \\
&\quad= \sum_{0\le n\le r+s}\left(\sum_{0\le k\le r}\binom{r}{k}_q\binom{s}{n-k}_q q^{k(k-1)/2}q^{(n-k)(n-k-1)/2}q^{r(n-k)}\right)x^n \\
&\quad= \sum_{0\le n\le r+s}\left(\sum_{0\le k\le r}\binom{r}{k}_q\binom{s}{n-k}_q q^{k^2/2-k/2+n^2/2-nk/2-kn/2+k^2/2-n/2+k/2+rn-rk}\right)x^n \\
&\quad= \sum_{0\le n\le r+s}\left(\sum_{0\le k\le r}\binom{r}{k}_q\binom{s}{n-k}_q q^{k^2+n^2/2-nk-n/2+rn-rk}\right)x^n.
\end{aligned}$$

Equating coefficients yields

$$\binom{r+s}{n}_q q^{n(n-1)/2} = \sum_{0\le k\le r}\binom{r}{k}_q\binom{s}{n-k}_q q^{k^2+n^2/2-nk-n/2+rn-rk}$$

if and only if

$$\begin{aligned}
\binom{r+s}{n}_q &= \sum_{0\le k\le r} \binom{r}{k}_q \binom{s}{n-k}_q q^{k^2+n^2/2-nk-n/2+rn-rk} \Big/ q^{n(n-1)/2} \\
&= \sum_{0\le k\le r} \binom{r}{k}_q \binom{s}{n-k}_q q^{k^2+n^2/2-nk-n/2+rn-rk-n^2/2+n/2} \\
&= \sum_{0\le k\le r} \binom{r}{k}_q \binom{s}{n-k}_q q^{k^2-nk+rn-rk} \\
&= \sum_{0\le k\le r} \binom{r}{k}_q \binom{s}{n-k}_q q^{(r-k)(n-k)} \\
&= \sum_{0\le k\le s} \binom{s}{k}_q \binom{r}{n-k}_q q^{(s-k)(n-k)} \\
&= \sum_{0\le k\le r} \binom{r}{k}_q \binom{s}{n-k}_q q^{(s-n+k)k}
\end{aligned}$$

so that the q-nomial generalizations of the fundamental identity (21) is

$$\binom{r+s}{n}_q = \sum_{0\le k\le r} \binom{r}{k}_q \binom{s}{n-k}_q q^{(r-k)(n-k)} = \sum_{0\le k\le r} \binom{r}{k}_q \binom{s}{n-k}_q q^{(s-n+k)k}.$$

H. A. Rothe, *Systematisches Lehrbuch der Arithmetik* (Leipzig: 1811), xxix; F. Schweins, *Analysis* (Heidelberg: 1820), §151; D. E. Knuth, *J. Combinatorial Theory* **A10** (1971), 178–180; G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge Univ. Press, 1990); C. F. Gauss, *Commentationes societatis regiæ scientiarum Gottingensis recentiores* **1** (1808), 147–186; Cauchy, *Comptes Rendus Acad. Sci.* **17** (Paris, 1843), 523–531; C. G. J. Jacobi, *Crelle* **32** (1846), 197–204; E. Heine, *Crelle* **34** (1847), 285–328.

59. [*M25*] A sequence of numbers A_{nk}, $n \ge 0$, $k \ge 0$, satisfies the relations $A_{n0} = 1$, $A_{0k} = \delta_{0k}$, $A_{nk} = A_{(n-1)k} + A_{(n-1)(k-1)} + \binom{n}{k}$ for $nk > 0$. Find A_{nk}.

We have that

$$A_{nk} = (n+1)\binom{n}{k} - \binom{n}{k+1}$$

and may prove this by induction.

Proposition. $A_{nk} = (n+1)\binom{n}{k} - \binom{n}{k+1}$.

Proof. Let n and k be arbitrary integers such that $n \ge 0$, $k \ge 0$, and $nk > 0$. We must show that

$$A_{nk} = (n+1)\binom{n}{k} - \binom{n}{k+1}.$$

First note that

$$A_{n0} = (n+1)\binom{n}{0} - \binom{n}{0+1} = n+1-n = 1$$

and

$$A_{0k} = (0+1)\binom{0}{k} - \binom{0}{k+1} = \delta_{0k}$$

as required. If $n = 1$, we have

$$\begin{aligned}
A_{1k} &= A_{0k} + A_{0(k-1)} + \binom{1}{k} \\
&= \delta_{0k} + \delta_{0(k-1)} + \binom{1}{k} \\
&= \binom{0}{k} - \binom{0}{k+1} + \binom{0}{k-1} - \binom{0}{k} + \binom{1}{k} \\
&= \binom{0}{k} + \binom{0}{k-1} + \binom{1}{k} - \left(\binom{0}{k+1} + \binom{0}{k}\right) \\
&= \binom{1}{k} + \binom{1}{k} - \binom{1}{k+1} \\
&= 2\binom{1}{k} - \binom{1}{k+1} \\
&= (1+1)\binom{1}{k} - \binom{1}{k+1}.
\end{aligned}$$

Then, assuming

$$A_{nk} = (n+1)\binom{n}{k} - \binom{n}{k+1}$$

we must show that

$$A_{(n+1)k} = ((n+1)+1)\binom{n+1}{k} - \binom{n+1}{k+1}.$$

But

$$\begin{aligned}
A_{(n+1)k} &= A_{nk} + A_{n(k-1)} + \binom{n+1}{k} \\
&= (n+1)\binom{n}{k} - \binom{n}{k+1} + (n+1)\binom{n}{k-1} - \binom{n}{k} + \binom{n+1}{k} \\
&= (n+1)\left(\binom{n}{k} + \binom{n}{k-1}\right) + \binom{n+1}{k} - \left(\binom{n}{k+1} + \binom{n}{k}\right) \\
&= (n+1)\binom{n+1}{k} + \binom{n+1}{k} - \binom{n+1}{k+1} \\
&= (n+2)\binom{n+1}{k} - \binom{n+1}{k+1} \\
&= ((n+1)+1)\binom{n+1}{k} - \binom{n+1}{k+1}
\end{aligned}$$

as we needed to show. □

60. [*M23*] We have seen that $\binom{n}{k}$ is the number of combinations of n things, k at a time, namely the number of ways to choose k different things out of a set of n. The *combinations with repetitions* are similar to ordinary combinations, except that we may choose each object any number of times. Thus, the list (1) would be extended to include also *aaa*, *aab*, *aac*, *aad*, *aae*, *abb*, etc., if we were considering combinations with repetition. How many k-combinations of n objects are there, if repetition is allowed?

The number of k-combinations of n objects, if repetition is allowed, is given by

$$\binom{n+k-1}{k}.$$

The number of k-combinations of n objects, if repetition is *not* allowed is simply the number of integer solutions $(o_1, o_2, \ldots, o_k)$ such that $0 < o_1 < o_2 < \cdots < o_k < n+1$, known to be

$$|\{o_i : 0 < o_1 < o_2 < \cdots < o_k < n+1\}| = \binom{n}{k}.$$

If repetitions *are* allowed, we want to determine

$$\begin{aligned}
&|\{o_i : 0 < o_1 \le o_2 \le \cdots \le o_k < n+1\}| \\
&\quad = |\{o_i : 0 < o_1 < o_2 + 1 < \cdots < o_k + k - 1 < n + k - 1 + 1\}| \\
&\quad = |\{o'_i : 0 < o'_1 < o'_2 < \cdots < o'_k < n + k\}| \\
&\quad = \binom{n+k-1}{k}
\end{aligned}$$

and hence the result.

H. F. Sherk, *Crelle* **3** (1828), 97; W. A. Förstemann, *Crelle* **13** (1835), 237.

61. [*M25*] Evaluate the sum

$$\sum_k \left[{n+1 \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m},$$

thereby obtaining a companion formula for Eq. (55).

We have

$$\begin{aligned}
&\sum_k \left[{n+1 \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} \\
&\quad = \sum_k \left(n\left[{n \atop k+1}\right] + \left[{n \atop k}\right] \right) \left\{{k \atop m}\right\} (-1)^{k-m} && \text{from Eq. (46)} \\
&\quad = n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} + \sum_k \left[{n \atop k}\right] \left\{{k \atop m}\right\} (-1)^{k-m} \\
&\quad = n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} + (-1)^{n-m} \sum_k \left[{n \atop k}\right] \left\{{k \atop m}\right\} (-1)^{n-k} \\
&\quad = n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} + (-1)^{n-m} \delta_{mn} && \text{from Eq. (47)} \\
&\quad = n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} + \delta_{mn}.
\end{aligned}$$

If $n < m$ then $k < m$ and

$$n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} + \delta_{mn} = 0 + 0 = 0.$$

If $n = m$ then $k < m$ and

$$n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} + \delta_{mn} = 0 + 1 = 1.$$

Otherwise, if $n > m$

$$n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} + \delta_{mn} = n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} = n!/m!$$

which we may prove by induction.

Proposition. $n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} = n!/m!$ *for* $0 \leq m < n$.

Proof. Let m and n be arbitrary nonnegative integers such that $m < n$. We must show that

$$n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} = n!/m!$$

If $n = 1$, $0 \leq m < n = 1 \Longrightarrow m = 0$ and

$$\begin{aligned} & n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} \\ &= \sum_k \left[{1 \atop k+1}\right] \left\{{k \atop 0}\right\} (-1)^{k} \\ &= \sum_k \left[{1 \atop k+1}\right] \left\{{k \atop 0}\right\} (-1)^{k} \\ &= \left[{1 \atop 1}\right] \left\{{0 \atop 0}\right\} (-1)^{0} \\ &= 1 \\ &= 1!/0! \\ &= n!/m! \end{aligned}$$

Then, assuming

$$n \sum_k \left[{n \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} = n!/m!$$

we must show that

$$(n+1) \sum_k \left[{n+1 \atop k+1}\right] \left\{{k \atop m}\right\} (-1)^{k-m} = (n+1)!/m!$$

But

$$\begin{aligned}
&(n+1)\sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} \\
&\quad = (n+1)\sum_k \left(n\begin{bmatrix} n \\ k+1 \end{bmatrix} + \begin{bmatrix} n \\ k \end{bmatrix} \right) \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} \\
&\quad = (n+1)\sum_k n\begin{bmatrix} n \\ k+1 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} + (n+1)\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} \\
&\quad = (n+1)n\sum_k \begin{bmatrix} n \\ k+1 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} + (n+1)\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} \\
&\quad = (n+1)n!/m! + (n+1)\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} \\
&\quad = (n+1)n!/m! + (n+1)\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{n-k}(-1)^{k-m-n+k} \\
&\quad = (n+1)n!/m! + (n+1)(-1)^{m+n}\sum_k \begin{bmatrix} n \\ k \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{n-k} \\
&\quad = (n+1)n!/m! + (n+1)(-1)^{m+n}\delta_{mn} && \text{from Eq. (47)} \\
&\quad = (n+1)n!/m! + 0 \\
&\quad = (n+1)n!/m! \\
&\quad = (n+1)!/m!
\end{aligned}$$

as we needed to show. □

And so, since

$$\sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} = \begin{cases} 0 & \text{if } n < m \\ 1 & \text{if } n = m \\ n!/m! & \text{otherwise} \end{cases}$$

we have that

$$\sum_k \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} \begin{Bmatrix} k \\ m \end{Bmatrix} (-1)^{k-m} = [n \geq m]n!/m!$$

62. [*M23*] The text gives formulas for sums involving a product of two binomial coefficients. Of the sums involving a product of three binomial coefficients, the following one and the identity of exercise 31 seem to be most useful:

$$\sum_k (-1)^k \binom{l+m}{l+k}\binom{m+n}{m+k}\binom{n+l}{n+k} = \frac{(l+m+n)!}{l!m!n!}, \qquad \text{integer } l,m,n \geq 0.$$

(The sum includes both positive and negative values of k.) Prove this identity. [*Hint:* There is a very short proof, which begins by applying the result of exercise 31.]

Proposition. $\sum_k (-1)^k \binom{l+m}{l+k}\binom{m+n}{m+k}\binom{n+l}{n+k} = \frac{(l+m+n)!}{l!m!n!}$, *integer* $l,m,n \geq 0$.

Proof. Let l,m,n be arbitrary nonnegative integers. We must show that

$$\sum_k (-1)^k \binom{l+m}{l+k}\binom{m+n}{m+k}\binom{n+l}{n+k} = \frac{(l+m+n)!}{l!m!n!}.$$

But from exercise 31

$$\sum_k \binom{m-r+s}{k}\binom{n+r-s}{n-k}\binom{r+k}{m+n} = \binom{r}{m}\binom{s}{n}$$

we have for $m' = m+k, n' = l-k, r' = m+n, s' = n+l, k' = j$ that

$$\begin{aligned}
&\sum_k (-1)^k \binom{l+m}{l+k}\binom{m+n}{m+k}\binom{n+l}{n+k} \\
&\quad = \sum_k (-1)^k \binom{l+m}{l+k}\binom{r'}{m'}\binom{s'}{s'-n'} \\
&\quad = \sum_k (-1)^k \binom{l+m}{l+k}\binom{r'}{m'}\binom{s'}{n'} \\
&\quad = \sum_k (-1)^k \binom{l+m}{l+k}\sum_{k'} \binom{m'-r'+s'}{k'}\binom{n'+r'-s'}{n'-k'}\binom{r'+k'}{m'+n'} \\
&\quad = \sum_k (-1)^k \binom{l+m}{l+k}\sum_{j} \binom{m+k-m-n+n+l}{j}\binom{l-k+m+n-n-l}{l-k-j}\binom{m+n+j}{m+k+l-k} \\
&\quad = \sum_{k,j} (-1)^k \binom{l+m}{l+k}\binom{l+k}{j}\binom{m-k}{l-k-j}\binom{m+n+j}{m+l} \\
&\quad = \sum_{k,j} (-1)^k \frac{(l+m)!}{(m-k)!(l+k)!}\frac{(l+k)!}{(l+k-j)!j!}\frac{(m-k)!}{(m-l+j)!(l-k-j)!}\frac{(m+n+j)!}{(n+j-l)!(m+l)!} \\
&\quad = \sum_{k,j} (-1)^k \frac{1}{(l+k-j)!j!}\frac{1}{(m-l+j)!(l-k-j)!}\frac{(m+n+j)!}{(n+j-l)!} \\
&\quad = \sum_{k,j} (-1)^k \frac{1}{(l-j-k)!(l-j+k)!}\frac{(m+n+j)!}{j!(m-l+j)!(n+j-l)!} \\
&\quad = \sum_{k,j} (-1)^k \frac{(2l-2j)!}{(2l-2j-(l-j+k))!(l-j+k)!(2l-2j)!}\frac{(m+n+j)!}{j!(m-l+j)!(n+j-l)!} \\
&\quad = \sum_{k,j} (-1)^k \binom{2l-2j}{l-j+k}\frac{(m+n+j)!}{(2l-2j)!j!(m-l+j)!(n+j-l)!} \\
&\quad = \sum_{k,j} (-1)^k \binom{2l-2j}{l-j+k}\frac{1}{(2(l-j))!}\frac{(m+n+j)!}{j!(m-l+j)!(n+j-l)!} \\
&\quad = \sum_{k,j} (-1)^k \binom{2l-2j}{l-j+k}(0-2(l-j))!\frac{0!}{(2(l-j))!(0-2(l-j))!}\frac{(m+n+j)!}{j!(m-l+j)!(n+j-l)!} \\
&\quad = \sum_{k,j} (-1)^k \binom{2l-2j}{l-j+k}(2(j-l))!\binom{0}{2(l-j)}\frac{(m+n+j)!}{j!(m-l+j)!(n+j-l)!} \\
&\quad = \sum_{k,j} (-1)^k \binom{2l-2j}{l-j+k}(2(j-l))!\delta_{lj}\frac{(m+n+j)!}{j!(m-l+j)!(n+j-l)!} \\
&\quad = \sum_{k} (-1)^k \binom{2l-2l}{l-l+k}(2(l-l))!\frac{(m+n+l)!}{l!(m-l+l)!(n+l-l)!} \\
&\quad = \sum_{k} (-1)^k \binom{0}{k}0!\frac{(m+n+l)!}{l!m!n!} \\
&\quad = (-1)^0\frac{(l+m+n)!}{l!m!n!} \\
&\quad = \frac{(l+m+n)!}{l!m!n!}
\end{aligned}$$

as we needed to show. □

A. C. Dixon, *Messenger of Math.* **20** (1891), 79–80; A. C. Dixon, *Proc. London Math. Soc.* **35** (1903), 285–289; L. J. Rogers, *Proc. London Math. Soc.* **26** (1895), 15–32, §8; P. A. MacMahon, *Quarterly Journal of Pure and Applied Math.* **33** (1902), 274–288; John Dougall, *Proc. Edinburgh Math. Society* **25** (1907), 114–132.

63. [*M30*] If l, m, and n are integers and $n \geq 0$, prove that

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}.$$

Proposition. *$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}$, integers l, m, and $n \geq 0$.*

Proof. Let l, m, and n be arbitrary integers such that $n \geq 0$. We must show that

$$\sum_{j,k} (-1)^{j+k} \binom{j+k}{k+l} \binom{r}{j} \binom{n}{k} \binom{s+n-j-k}{m-j} = (-1)^l \binom{n+r}{n+l} \binom{s-r}{m-n-l}.$$

But as a polynomial in arbitary reals x and y, we have

$$
\begin{aligned}
&\sum_{l,m}\sum_{j,k}(-1)^{j+k}\binom{j+k}{l+k}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}x^l y^m \\
&\quad= \sum_{m}\sum_{j,k}(-1)^{j+k}\sum_{l}\binom{j+k}{l+k}x^l\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}y^m \\
&\quad= \sum_{m}\sum_{j,k}(-1)^{j+k}\sum_{l+k}\binom{j+k}{l+k}x^l\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}y^m \\
&\quad= \sum_{m}\sum_{j,k}(-1)^{j+k}\sum_{l+k}\binom{j+k}{l+k}\frac{x^{l+k}}{x^k}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}y^m \\
&\quad= \sum_{m}\sum_{j,k}(-1)^{j+k}\frac{(1+x)^{j+k}}{x^k}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j}y^m \\
&\quad= \sum_{j,k}(-1)^{j+k}\frac{(1+x)^{j+k}}{x^k}\binom{r}{j}\binom{n}{k}\sum_{m}\binom{s+n-j-k}{m-j}y^m \\
&\quad= \sum_{j,k}(-1)^{j+k}\frac{(1+x)^{j+k}}{x^k}\binom{r}{j}\binom{n}{k}\sum_{m-j}\binom{s+n-j-k}{m-j}y^m \\
&\quad= \sum_{j,k}(-1)^{j+k}\frac{(1+x)^{j+k}}{x^k}\binom{r}{j}\binom{n}{k}\sum_{m-j}\binom{s+n-j-k}{m-j}y^{m-j}y^j \\
&\quad= \sum_{j,k}(-1)^{j+k}\frac{(1+x)^{j+k}}{x^k}\binom{r}{j}\binom{n}{k}(1+y)^{s+n-j-k}y^j \\
&\quad= \sum_{j}\binom{r}{j}(-1)^j\frac{(1+x)^j y^j}{(1+y)^j}\sum_{k}\binom{n}{k}(-1)^k\frac{(1+x)^k}{(1+y)^k x^k}(1+y)^{s+n} \\
&\quad= \sum_{j}\binom{r}{j}\left(-\frac{(1+x)y}{1+y}\right)^j\sum_{k}\binom{n}{k}\left(-\frac{1+x}{(1+y)x}\right)^k(1+y)^{s+n} \\
&\quad= \left(1-\frac{(1+x)y}{1+y}\right)^r\left(1-\frac{1+x}{(1+y)x}\right)^n(1+y)^{s+n} \\
&\quad= \left(\frac{1+y-(1+x)y}{1+y}\right)^r\left(\frac{(1+y)x-(1+x)}{(1+y)x}\right)^n(1+y)^{s+n} \\
&\quad= \left(\frac{1-xy}{1+y}\right)^r\left(\frac{(-1)(1-xy)}{(1+y)x}\right)^n(1+y)^{s+n} \\
&\quad= \frac{(1-xy)^r}{(1+y)^r}\frac{(-1)^n(1-xy)^n}{(1+y)^n x^n}(1+y)^{s+n} \\
&\quad= \frac{(1-xy)^r}{(1+y)^r}\frac{(-1)^n(1-xy)^n}{x^n}(1+y)^{s} \\
&\quad= \frac{(-1)^n(1-xy)^{n+r}(1+y)^{s-r}}{x^n}.
\end{aligned}
$$

Continuing,

$$\begin{aligned}
&\frac{(-1)^n(1-xy)^{n+r}(1+y)^{s-r}}{x^n}\\
&\quad= \frac{(-1)^n\sum_{n+l}\binom{n+r}{n+l}(-xy)^{n+l}(1+y)^{s-r}}{x^n}\\
&\quad= \frac{(-1)^n\sum_{l}\binom{n+r}{n+l}(-xy)^{n+l}(1+y)^{s-r}}{x^n}\\
&\quad= \sum_l (-1)^{2n+l}\binom{n+r}{n+l}(1+y)^{s-r}x^l y^{n+l}\\
&\quad= \sum_l (-1)^{l}\binom{n+r}{n+l}(1+y)^{s-r}x^l y^{n+l}\\
&\quad= \sum_l (-1)^{l}\binom{n+r}{n+l}\sum_{m-n-l}\binom{s-r}{m-n-l}y^{m-n-l}x^l y^{n+l}\\
&\quad= \sum_l (-1)^{l}\binom{n+r}{n+l}\sum_{m}\binom{s-r}{m-n-l}y^{m-n-l}x^l y^{n+l}\\
&\quad= \sum_{l,m} (-1)^{l}\binom{n+r}{n+l}\binom{s-r}{m-n-l}x^l y^{m}.
\end{aligned}$$

Equating coefficients yields the result

$$\sum_{j,k}(-1)^{j+k}\binom{j+k}{k+l}\binom{r}{j}\binom{n}{k}\binom{s+n-j-k}{m-j} = (-1)^l\binom{n+r}{n+l}\binom{s-r}{m-n-l}$$

as we needed to show. □

CMath, exercises 5.83 and 5.106.

▶ **64.** [*M20*] Show that $\left\{ {n \atop m} \right\}$ is the number of ways to partition a set of n elements into m nonempty disjoint subsets. For example, the set $\{1,2,3,4\}$ can be partitioned into two subsets in $\left\{ {4 \atop 2} \right\} = 7$ ways: $\{1,2,3\}\{4\}$; $\{1,2,4\}\{3\}$; $\{1,3,4\}\{2\}$; $\{2,3,4\}\{1\}$; $\{1,2\}\{3,4\}$; $\{1,3\}\{2,4\}$; $\{1,4\}\{2,3\}$. *Hint:* Use Eq. (46).

Let $p(n,m)$ denote the number of ways to partition a set of n elements into m nonempty disjoint subsets for nonnegative integers m, n. If $n = 0$, then clearly

$$p(0,m) = \delta_{0m} = \left\{ {0 \atop m} \right\}.$$

Otherwise, if $n > 0$, we seek the number of partitions which contain the set n, given by $p(n-1, m-1)$ and the number of partitions in which n has been inserted into sets with other elements, given by $mp(n-1,m)$. That is, from Eq. (46) and induction,

$$p(n,m) = p(n-1,m-1) + mp(n-1,m) = \left\{ {n \atop m} \right\},$$

and hence the claim.

65. [*HM35*] (B. F. Logan.) Prove Eqs. (59) and (60).

We may prove Eq. (59).

Proposition. $z^r = \sum_k \left\{ {r \atop r-k} \right\} z^{\underline{r-k}}$ *for* $\mathrm{Re}(z) > 0$.

Proof. Let r, z be arbitrary complex numbers such that $\operatorname{Re}(z) > 0$. We must show that

$$z^r = \sum_k \left\{ {r \atop r-k} \right\} z^{\underline{r-k}}.$$

In the case that $\operatorname{Re}(r) < 1$, by definition and since $\operatorname{Re}(z) > 0$,

$$\begin{aligned}
\Gamma(1-r) &= \int_0^\infty e^{-t} t^{1-r-1} dt \\
&= \int_0^\infty e^{-t} t^{-r} dt \\
&= \frac{z^{r-1}}{z^{r-1}} \int_0^\infty e^{-t} t^{-r} dt \\
&= \frac{1}{z^{r-1}} \int_0^\infty z^{r-1} e^{-t} t^{-r} dt \\
&= \frac{1}{z^{r-1}} \int_0^\infty z^{r-1} e^{-zt} (zt)^{-r} dzt \\
&= \frac{1}{z^{r-1}} \int_0^\infty z^{r-1-r+1} e^{-zt} t^{-r} dt \\
&= \frac{1}{z^{r-1}} \int_0^\infty e^{-zt} t^{-r} dt
\end{aligned}$$

if and only if

$$\begin{aligned}
z^r &= \frac{z}{\Gamma(1-r)} \int_0^\infty e^{-zt} t^{-r} dt \\
&= \frac{z}{\Gamma(1-r)} \int_0^1 (1-u)^z (-\ln(1-u))^{-r} d(-\ln(1-u)) \qquad \text{for } e^{-t} = 1-u \\
&= \frac{z}{\Gamma(1-r)} \int_0^1 (1-u)^{z-1} \left(\ln\left(\frac{1}{1-u} \right) \right)^{-r} du \\
&= \frac{z}{\Gamma(1-r)} \int_0^1 (1-u)^{z-1} \frac{u^{-r}}{u^{-r}} \left(\ln\left(\frac{1}{1-u} \right) \right)^{-r} du \\
&= \frac{z}{\Gamma(1-r)} \int_0^1 (1-u)^{z-1} u^{-r} \left(\frac{1}{u} \ln\left(\frac{1}{1-u} \right) \right)^{-r} du.
\end{aligned}$$

From Eq. (6.51)[1]

$$\begin{aligned}
\left(\frac{1}{u} \ln\left(\frac{1}{1-u} \right) \right)^{-r} &= -r \sum_k \left[{-r+k \atop -r} \right] \frac{u^k}{(-r+k)^{\underline{k+1}}} \\
&= -r \sum_k \left\{ {r \atop r-k} \right\} \frac{u^k}{(-r+k)^{\underline{k+1}}} \\
&= -r \sum_k \left\{ {r \atop r-k} \right\} \frac{\Gamma(-r) u^k}{\Gamma(-r+k+1)} \\
&= \sum_k \left\{ {r \atop r-k} \right\} \frac{\Gamma(1-r) u^k}{\Gamma(-r+k+1)}
\end{aligned}$$

[1]Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics*, 2nd edition (Reading, Mass.: Addison-Wesley, 1994), 272.

and given the definition of the β function as

$$\int_0^\infty (1-u)^{z-1}u^{k-r}du = \beta(k-r+1,z) = \frac{\Gamma(1-r+k)\Gamma(z)}{\Gamma(1-r+k+z)}$$

we have that

$$\begin{aligned}
z^r &= \frac{z}{\Gamma(1-r)}\int_0^1 (1-u)^{z-1}u^{-r}\left(\frac{1}{u}\ln\left(\frac{1}{1-u}\right)\right)^{-r}du \\
&= \frac{z}{\Gamma(1-r)}\int_0^1 (1-u)^{z-1}u^{-r}\sum_k \left\{ r \atop r-k \right\}\frac{\Gamma(1-r)u^k}{\Gamma(-r+k+1)}du \\
&= \sum_k \left\{ r \atop r-k \right\}\frac{z}{\Gamma(-r+k+1)}\int_0^1 (1-u)^{z-1}u^{k-r}du \\
&= \sum_k \left\{ r \atop r-k \right\}\frac{z}{\Gamma(-r+k+1)}\frac{\Gamma(1-r+k)\Gamma(z)}{\Gamma(1-r+k+z)} \\
&= \sum_k \left\{ r \atop r-k \right\}\frac{z\Gamma(z)}{\Gamma(1-r+k+z)} \\
&= \sum_k \left\{ r \atop r-k \right\}\frac{\Gamma(z+1)}{\Gamma(z-r+k+1)} \\
&= \sum_k \left\{ r \atop r-k \right\}\frac{z!}{(z-r+k)!} \\
&= \sum_k \left\{ r \atop r-k \right\}z^{\underline{r-k}}
\end{aligned}$$

which establishes the case $\mathrm{Re}(r) < 1$. Then, assuming

$$z^r = \sum_k \left\{ r \atop r-k \right\}z^{\underline{r-k}}$$

we must show that

$$z^{r+1} = \sum_k \left\{ r+1 \atop r+1-k \right\}z^{\underline{r+1-k}}.$$

But from the recurrence relations for falling factorial powers

$$zz^{\underline{r-k}} = z^{\underline{r-k+1}} + (r-k)z^{\underline{r-k}}$$

and Eq. (46) we have that

$$\begin{aligned}
z^{\underline{r+1}} &= z z^{\underline{r}} \\
&= z \sum_k \left\{ {r \atop r-k} \right\} z^{\underline{r-k}} \\
&= \sum_k \left\{ {r \atop r-k} \right\} z z^{\underline{r-k}} \\
&= \sum_k \left\{ {r \atop r-k} \right\} (z^{\underline{r-k+1}} + (r-k) z^{\underline{r-k}}) \\
&= \sum_k \left\{ {r \atop r-k} \right\} z^{\underline{r-k+1}} + \sum_{k-1} \left\{ {r \atop r-(k-1)} \right\} (r-(k-1)) z^{\underline{r-(k-1)}} \\
&= \sum_k \left\{ {r \atop r-k} \right\} z^{\underline{r-k+1}} + \sum_k \left\{ {r \atop r-k+1} \right\} (r-k+1) z^{\underline{r-k+1}} \\
&= \sum_k \left(\left\{ {r \atop r-k} \right\} + (r-k+1) \left\{ {r \atop r-k+1} \right\} \right) z^{\underline{r-k+1}} \\
&= \sum_k \left\{ {r+1 \atop r+1-k} \right\} z^{\underline{r+1-k}}
\end{aligned}$$

as we needed to show. □

We may also prove Eq. (60).

Proposition. $z^{\underline{r}} = \sum_{0 \le k \le m} \left[{r \atop r-k} \right] (-1)^k z^{r-k} + O(z^{r-m-1})$.

Proof. Let r, z be arbitrary complex numbers. We must show that

$$z^{\underline{r}} = \sum_{0 \le k \le m} \left[{r \atop r-k} \right] (-1)^k z^{r-k} + O(z^{r-m-1}).$$

By Euler-Maclaurin summation for Stirling's approximation[2] we have that

$$\begin{aligned}
\sum_{1 \le k < z} \ln(k) &= z \ln(z) - z + \sigma - \frac{\ln(z)}{2} \\
&\quad + \sum_{1 \le k \le m} \frac{B_{2k}}{2k(2k-1) z^{2k-1}} \\
&\quad + \varphi_{m,z} \frac{B_{2m+2}}{(2m+2)(2m+1) z^{2m+1}}, \\
\sum_{1 \le k < z-r} \ln(k) &= (z-r) \ln(z-r) - (z-r) + \sigma - \frac{\ln(z-r)}{2} \\
&\quad + \sum_{1 \le k \le m} \frac{B_{2k}}{2k(2k-1)(z-r)^{2k-1}} \\
&\quad + \varphi_{m,z-r} \frac{B_{2m+2}}{(2m+2)(2m+1)(z-r)^{2m+1}}
\end{aligned}$$

for an arbitrary positive integer m, constant σ, "Stirling's constant," Bernoulli numbers

[2] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, *Concrete Mathematics*, 2nd edition (Reading, Mass.: Addison-Wesley, 1994), 481.

B_k, and $0 < \varphi_{m,x} < 1$, so that

$$
\begin{aligned}
&\ln \frac{z^{\underline{r}}}{z^r} \\
&\quad = \ln \frac{z!}{z^r (z-r)!} \\
&\quad = \ln(z!) - \ln((z-r)!) - r\ln(z) \\
&\quad = -r\ln(z) + \sum_{1\le k\le z} \ln(k) - \sum_{1\le k\le z-r} \ln(k) \\
&\quad = -r\ln(z) + \ln(z) - \ln(z-r) + \sum_{1\le k< z} \ln(k) - \sum_{1\le k< z-r} \ln(k) \\
&\quad = (1-r)\ln(z) - \ln(z-r) + \sum_{1\le k< z} \ln(k) - \sum_{1\le k< z-r} \ln(k) \\
&\quad = (1-r)\ln(z) - \ln(z-r) \\
&\qquad + z\ln(z) - z + \sigma - \frac{\ln(z)}{2} \\
&\qquad + \sum_{1\le k\le m} \frac{B_{2k}}{2k(2k-1)z^{2k-1}} \\
&\qquad + \varphi_{m,z} \frac{B_{2m+2}}{(2m+2)(2m+1)z^{2m+1}} \\
&\qquad - (z-r)\ln(z-r) + (z-r) - \sigma + \frac{\ln(z-r)}{2} \\
&\qquad - \sum_{1\le k\le m} \frac{B_{2k}}{2k(2k-1)(z-r)^{2k-1}} \\
&\qquad - \varphi_{m,z-r} \frac{B_{2m+2}}{(2m+2)(2m+1)(z-r)^{2m+1}} \\
&\quad = -r + (z-r+1/2)\ln(z) - (z-r+1/2)\ln(z-r) \\
&\qquad + \sum_{1\le k\le m} \frac{B_{2k}}{2k(2k-1)} \left(z^{-(2k-1)} - (z-r)^{-(2k-1)} \right) \\
&\qquad + \frac{B_{2m+2}}{(2m+2)(2m+1)} \left(\varphi_{m,z} z^{-(2m+1)} - \varphi_{m,z-r}(z-r)^{-(2m+1)} \right) \\
&\quad = -r - (z-r+1/2)\ln(1-r/z) \\
&\qquad + \sum_{1\le k\le m} \frac{B_{2k}}{2k(2k-1)} \left(z^{-(2k-1)} - (z-r)^{-(2k-1)} \right) \\
&\qquad + \frac{B_{2m+2}}{(2m+2)(2m+1)} \left(\varphi_{m,z} z^{-(2m+1)} - \varphi_{m,z-r}(z-r)^{-(2m+1)} \right).
\end{aligned}
$$

That is, so that $\ln \frac{z^{\underline{r}}}{z^r}$ is a series in which each coefficient of z^{-k} is a polynomial in r; and so similarly for the exponential

$$\frac{z^{\underline{r}}}{z^r} = \sum_{0\le k\le m} c_k(r) z^{-k} + O(z^{-m-1})$$

with coefficients $c_k(r)$, polynomials in r, and with asymptotic bounds $O(z^{-m-1})$, if and only if

$$z^{\underline{r}} = \sum_{0\le k\le m} c_k(r) z^{r-k} + O(z^{r-m-1}).$$

From Eq. (44) for r restricted to the integers

$$z^{\underline{r}} = \sum_{0 \le k \le r} \left[{r \atop k}\right] (-1)^{r-k} z^k = \sum_{0 \le k \le r} \left[{r \atop r-k}\right] (-1)^k z^{r-k},$$

and since $\left[{r \atop r-k}\right]$ is a polynomial in r of degree $2k$ whenever k is a nonnegative integer, the coefficients of z^{r-k} hold for arbitrary complex r such that

$$c_k(r) = \left[{r \atop r-k}\right] (-1)^k.$$

Therefore

$$\sum_{0 \le k \le m} c_k(r) z^{r-k} + O(z^{r-m-1})$$
$$= \sum_{0 \le k \le m} \left[{r \atop r-k}\right] (-1)^k z^{r-k} + O(z^{r-m-1}).$$

and hence the result

$$z^{\underline{r}} = \sum_{0 \le k \le m} \left[{r \atop r-k}\right] (-1)^k z^{r-k} + O(z^{r-m-1}).$$

□

AMM **99** (1992), 410–422.

66. [*HM30*] Suppose x, y, and z are real numbers satisfying

$$\binom{x}{n} = \binom{y}{n} + \binom{z}{n-1},$$

where $x \ge n-1$, $y \ge n-1$, $z > n-2$, and n is an integer ≥ 2. Prove that

$$\binom{x}{n-1} \le \binom{y}{n-1} + \binom{z}{n-2} \qquad \text{if and only if} \qquad y \ge z;$$
$$\binom{x}{n+1} \le \binom{y}{n+1} + \binom{z}{n} \qquad \text{if and only if} \qquad y \le z.$$

Proposition. $\binom{x}{n-1} \le \binom{y}{n-1} + \binom{z}{n-2}$ *iff* $y \ge z$, $\binom{x}{n+1} \le \binom{y}{n+1} + \binom{z}{n}$ *iff* $y \le z$.

Proof. Let x, y, and z be arbitrary real numbers and n an arbitrary integer ≥ 2 such that

$$\binom{x}{n} = \binom{y}{n} + \binom{z}{n-1},$$

$x \ge n-1$, $y \ge n-1$, and $z > n-2$. We must show that both

$$\binom{x}{n-1} \le \binom{y}{n-1} + \binom{z}{n-2} \iff y \ge z$$

and

$$\binom{x}{n+1} \le \binom{y}{n+1} + \binom{z}{n} \iff y \le z.$$

We will first show the former. Since $y \geq z$,

$$\begin{aligned}\binom{x}{n} &= \binom{y}{n} + \binom{z}{n-1}\\ &\geq \binom{z}{n} + \binom{z}{n-1}\\ &\geq \binom{z+1}{n},\end{aligned}$$

and so $x \geq z+1$. Given the identities

$$\begin{aligned}\binom{a+n-1}{n-1} &= \binom{a+0-(-1)(n-1)}{n-1}\\ &= \sum_{0\leq j\leq n-1} \binom{a-(-1)j}{j}\binom{0-(-1)((n-1)-j)}{(n-1)-j}\frac{a}{a-(-1)j} && \text{from Eq. (26)}\\ &= \sum_{0\leq j\leq n-1} \binom{a+j}{j}\binom{n-1-j}{n-1-j}\frac{a}{a+j}\\ &= \sum_{0\leq j\leq n-1} \binom{a+j}{j}\frac{a}{a+j}\\ &= \sum_{0\leq j\leq n-1} \frac{(a+j)!}{j!(a+j-j)!}\frac{a}{a+j}\\ &= \sum_{0\leq j\leq n-1} \frac{a(a+j)!}{(a+j)j!a!}\\ &= \sum_{0\leq j\leq n-1} \frac{(a+j-1)!}{j!(a+j-1-j)!}\\ &= \sum_{0\leq j\leq n-1} \binom{a+j-1}{j}\end{aligned}$$

and

$$\begin{aligned}\binom{a+b}{n} &= \binom{a+(b-n)-(-1)n}{n}\\ &= \sum_{0\leq j\leq n} \binom{a-(-1)j}{j}\binom{(b-n)-(-1)(n-j)}{n-j}\frac{a}{a-(-1)j} && \text{from Eq. (26)}\\ &= \sum_{0\leq j\leq n} \binom{a+j}{j}\binom{b-j}{n-j}\frac{a}{a+j}\\ &= \sum_{0\leq j\leq n} \frac{a+j-j}{a+j}\binom{a+j}{j}\binom{b-j}{n-j}\\ &= \sum_{0\leq j\leq n} \binom{a+j-1}{j}\binom{b-j}{n-j} && \text{from Eq. (8)}\end{aligned}$$

for arbitrary integers a, b; and letting $x = t + x'$, $y = t + y'$, $z = t + n - 1$,

$$\binom{x}{n} = \binom{t+x'}{n} = \sum_{0\le j\le n} \binom{t+j-1}{j}\binom{x'-j}{n-j},$$

$$\binom{y}{n} = \binom{t+y'}{n} = \sum_{0\le j\le n} \binom{t+j-1}{j}\binom{y'-j}{n-j},$$

$$\binom{z}{n-1} = \binom{t+n-1}{n-1} = \sum_{0\le j\le n-1} \binom{t+j-1}{j};$$

we have that

$$\begin{aligned}
0 &= \binom{y}{n} + \binom{z}{n-1} - \binom{x}{n} \\
&= \sum_{0\le j\le n} \binom{t+j-1}{j}\binom{y'-j}{n-j} + \sum_{0\le j\le n-1} \binom{t+j-1}{j} - \sum_{0\le j\le n} \binom{t+j-1}{j}\binom{x'-j}{n-j} \\
&= \binom{t+n-1}{n}\binom{y'-n}{n-n} - \binom{t+n-1}{n}\binom{x'-n}{n-n} \\
&\qquad + \sum_{0\le j\le n-1} \binom{t+j-1}{j}\left(\binom{y'-j}{n-j} + 1 - \binom{x'-j}{n-j}\right) \\
&= \binom{t+n-1}{n} - \binom{t+n-1}{n} + \sum_{0\le j\le n-1} \binom{t+j-1}{j}\left(\binom{y'-j}{n-j} + 1 - \binom{x'-j}{n-j}\right) \\
&= \sum_{0\le j\le n-1} \binom{t+j-1}{j}\varphi_{n-j}
\end{aligned}$$

where

$$\varphi_i = \binom{y'-n+i}{i} + 1 - \binom{x'-n+i}{i}.$$

Similarly, since t increases by one as n decreases by one,

$$\begin{aligned}
&\binom{y}{n-1} + \binom{z}{n-2} - \binom{x}{n-1} \\
&\quad = \sum_{0\le j\le n-2} \binom{t+1+j-1}{j}\varphi_{n-1-j} \\
&\quad = \sum_{1\le j\le n-1} \binom{t+j-1}{j-1}\varphi_{n-j} \\
&\quad = \sum_{1\le j\le n-1} \binom{t+j-1}{j-1}\varphi_{n-j} \\
&\quad = \sum_{1\le j\le n-1} \frac{j}{t+j-1-j+1}\binom{t+j-1}{j}\varphi_{n-j} && \text{from Eqs. (7), (8)} \\
&\quad = \sum_{1\le j\le n-1} \frac{j}{t}\binom{t+j-1}{j}\varphi_{n-j} \\
&\quad = \sum_{0\le j\le n-1} \frac{j}{t}\binom{t+j-1}{j}\varphi_{n-j}
\end{aligned}$$

with the understanding that in the case $t = 0$, we define $\frac{0}{t} = 0$. Then, for all i, $1 \le i \le n-1$, since $x \ge y \iff x' \ge y'$ by hypothesis and $x' = x - t \ge y - t \ge z + 1 - t = n$,

$$\begin{aligned}
\varphi_{i-1} &= \binom{y' - n + i - 1}{i - 1} + 1 - \binom{x' - n + i - 1}{i - 1} \\
&= \frac{i}{y' - n + i}\binom{y' - n + i}{i} + 1 - \frac{i}{x' - n + i}\binom{x' - n + i}{i} \\
&\ge \frac{i}{x' - n + i}\binom{y' - n + i}{i} + \frac{i}{x' - n + i} - \frac{i}{x' - n + i}\binom{x' - n + i}{i} \\
&= \frac{i}{x' - n + i}\left(\binom{y' - n + i}{i} + 1 - \binom{x' - n + i}{i}\right) \\
&= \frac{i}{x' - n + i}\varphi_i \\
&\ge 0.
\end{aligned}$$

but

$$\begin{aligned}
\varphi_n &= \binom{y' - n + n}{n} + 1 - \binom{x' - n + n}{n} \\
&= \binom{y'}{n} + 1 - \binom{x'}{n} \\
&= \binom{y'}{n} + \binom{n-1}{n-1} - \binom{x'}{n} \\
&= \binom{y - t}{n} + \binom{z - t}{n - 1} - \binom{x - t}{n} \\
&\le 0.
\end{aligned}$$

And so finally,

$$\begin{aligned}
&\binom{y}{n-1} + \binom{z}{n-2} - \binom{x}{n-1} \\
&\quad = \sum_{0 \le j \le n-1} \frac{j}{t}\binom{t + j - 1}{j}\varphi_{n-j} \\
&\quad \ge \sum_{0 \le j \le n-1} \frac{1}{t}\binom{t + j - 1}{j}\varphi_{n-j} \\
&\quad = \frac{1}{t}\sum_{0 \le j \le n-1} \binom{t + j - 1}{j}\varphi_{n-j} \\
&\quad = 0
\end{aligned}$$

and hence the former result.

We will then show the latter, and it is sufficient to show that if

$$\binom{x}{n+1} - \binom{y}{n+1} - \binom{z}{n} = 0$$

implies

$$\frac{d}{dz}\left(\binom{x}{n+1} - \binom{y}{n+1} - \binom{z}{n}\right) \le 0,$$

then

$$\binom{x}{n+1} \le \binom{y}{n+1} + \binom{z}{n} \quad \iff \quad y \le z.$$

But assuming

$$
\begin{aligned}
&\binom{x}{n+1} - \binom{y}{n+1} - \binom{z}{n} \\
&= \frac{x-n}{n+1}\binom{x}{n} - \frac{y-n}{n+1}\binom{y}{n} - \frac{z-(n-1)}{n}\binom{z}{n-1} \\
&= \frac{x-n}{n+1}\left(\binom{y}{n} + \binom{z}{n-1}\right) - \frac{y-n}{n+1}\binom{y}{n} - \frac{z-n+1}{n}\binom{z}{n-1} \\
&= \frac{x-n}{n+1}\binom{y}{n} + \frac{x-n}{n+1}\binom{z}{n-1} - \frac{y-n}{n+1}\binom{y}{n} - \frac{z-n+1}{n}\binom{z}{n-1} \\
&= \frac{x-n}{n+1}\binom{z}{n-1} + \frac{x-y}{n+1}\binom{y}{n} - \frac{z-n+1}{n}\binom{z}{n-1} \\
&= 0
\end{aligned}
$$

and since $x \geq y \iff x - y \geq 0$ by hypothesis, we have that

$$
\begin{aligned}
&\frac{x-n}{n+1}\binom{z}{n-1} + \frac{x-y}{n+1}\binom{y}{n} - \frac{z-n+1}{n}\binom{z}{n-1} = 0 \\
\iff \quad & \frac{x-n}{n+1}\binom{z}{n-1} - \frac{z-n+1}{n}\binom{z}{n-1} \leq 0 \\
\iff \quad & \frac{x-n}{n+1}\binom{z}{n-1} \leq \frac{z-n+1}{n}\binom{z}{n-1} \\
\iff \quad & \frac{x-n}{n+1} \leq \frac{z-n+1}{n}.
\end{aligned}
$$

Also, with $\frac{d}{dz}\binom{z}{n-1} = \frac{d}{dz}\binom{x}{n}$, and given that $\frac{d}{dn}n! = \frac{d}{dn}\Gamma(n+1) = n!\left(-\gamma + \sum_{1\leq k\leq n}\frac{1}{k}\right)$ for the Euler-Mascheroni constant γ,

$$
\begin{aligned}
&\frac{d}{dx}\binom{x}{n}\Big/\binom{x}{n} \\
&= \frac{1}{n!}\frac{d}{dx}\frac{x!}{(x-n)!}\Big/\frac{x!}{n!(x-n)!} \\
&= \frac{1}{n!}\frac{(x-n)!\frac{d}{dx}x! - x!\frac{d}{dx}(x-n)!}{((x-n)!)^2}\Big/\frac{x!}{n!(x-n)!} \\
&= \frac{(x-n)!\frac{d}{dx}x! - x!\frac{d}{dx}(x-n)!}{x!(x-n)!} \\
&= \frac{(x-n)!x!\left(-\gamma + \sum_{1\leq k\leq x}\frac{1}{k}\right) - x!(x-n)!\left(-\gamma + \sum_{1\leq k\leq x-n}\frac{1}{k}\right)}{x!(x-n)!} \\
&= -\gamma + \sum_{1\leq k\leq x}\frac{1}{k} + \gamma - \sum_{1\leq k\leq x-n}\frac{1}{k} \\
&= \sum_{x-n+1\leq k\leq x}\frac{1}{k} \\
&= \sum_{0\leq k\leq n-1}\frac{1}{x-k}
\end{aligned}
$$

and

$$\begin{aligned}
&\frac{n}{n+1}\frac{d}{dz}\binom{z}{n-1}\Big/\binom{z}{n-1} \\
&\quad= \frac{n}{(n+1)(n-1)!}\frac{d}{dz}\frac{z!}{(z-(n-1))!}\Big/\frac{z!}{(n-1)!(z-(n-1))!} \\
&\quad= \frac{n}{(n+1)(n-1)!}\frac{(z-(n-1))!\frac{d}{dz}z! - z!\frac{d}{dz}(z-(n-1))!}{((z-(n-1))!)^2}\Big/\frac{z!}{(n-1)!(z-(n-1))!} \\
&\quad= \frac{n}{n+1}\frac{(z-(n-1))!\frac{d}{dz}z! - z!\frac{d}{dz}(z-(n-1))!}{z!(z-(n-1))!} \\
&\quad= \frac{n}{n+1}\frac{(z-(n-1))!z!\left(-\gamma+\sum_{1\le k\le z}\frac{1}{k}\right) - z!(z-(n-1))!\left(-\gamma+\sum_{1\le k\le z-(n-1)}\frac{1}{k}\right)}{z!(z-(n-1))!} \\
&\quad= \frac{n}{n+1}\left(-\gamma+\sum_{1\le k\le z}\frac{1}{k}+\gamma-\sum_{1\le k\le z-(n-1)}\frac{1}{k}\right) \\
&\quad= \frac{n}{n+1}\sum_{z-(n-1)+1\le k\le z}\frac{1}{k} \\
&\quad= \frac{n}{n+1}\sum_{0\le k\le n-2}\frac{1}{z-k}.
\end{aligned}$$

Also, since

$$\begin{aligned}
&\iff \quad 1 = 1 \\
&\iff \quad \frac{d}{dx}\binom{x}{n}\Big/\frac{d}{dx}\binom{x}{n} = \frac{d}{dz}\binom{z}{n-1}\Big/\frac{d}{dz}\binom{z}{n-1} \\
&\iff \quad \frac{d}{dx}\binom{x}{n}\frac{d}{dz}\binom{z}{n-1}\Big/\frac{d}{dx}\binom{x}{n} = \frac{d}{dz}\binom{z}{n-1} \\
&\iff \quad \frac{d}{dx}\binom{x}{n}\frac{dx}{dz} = \frac{d}{dz}\binom{z}{n-1}
\end{aligned}$$

and since for arbitrary integers $k \geq 0$, $n \leq n+1 \Longrightarrow \frac{k}{n+1} \leq \frac{k}{n}$, we have that

$$
\begin{aligned}
&\frac{x-n}{n+1} \leq \frac{z-n+1}{n} \quad \wedge \quad \frac{k}{n+1} \leq \frac{k}{n} \\
&\quad\Longrightarrow \quad \frac{x-n+k}{n+1} + \frac{k}{n+1} \leq \frac{z-n+1}{n} + \frac{k}{n}. \\
&\quad\Longleftrightarrow \quad \frac{x-n+k}{n+1} \leq \frac{z-n+1+k}{n} \\
&\quad\Longleftrightarrow \quad \frac{1}{x-n+k} \geq \frac{n}{n+1} \frac{1}{z-n+1+k} \\
&\quad\Longrightarrow \quad \sum_{0 \leq k \leq n-1} \frac{1}{x-k} \geq \frac{n}{n+1} \sum_{0 \leq k \leq n-1} \frac{1}{z-k} \\
&\quad\Longrightarrow \quad \sum_{0 \leq k \leq n-1} \frac{1}{x-k} \geq \frac{n}{n+1} \sum_{0 \leq k \leq n-2} \frac{1}{z-k} \\
&\quad\Longrightarrow \quad \frac{d}{dx}\binom{x}{n} \Big/ \binom{x}{n} \geq \frac{n}{n+1} \frac{d}{dz}\binom{z}{n-1} \Big/ \binom{z}{n-1} \\
&\quad\Longleftrightarrow \quad \binom{z}{n-1} \geq \frac{n}{n+1}\binom{x}{n} \frac{d}{dz}\binom{z}{n-1} \Big/ \frac{d}{dx}\binom{x}{n} \\
&\quad\Longleftrightarrow \quad \binom{z}{n-1} \geq \frac{n}{n+1}\binom{x}{n}\frac{dx}{dz} \\
&\quad\Longleftrightarrow \quad \frac{1}{n}\binom{z}{n-1} \geq \frac{1}{n+1}\binom{x}{n}\frac{dx}{dz} \\
&\quad\Longleftrightarrow \quad \frac{1}{n+1}\binom{x}{n}\frac{dx}{dz} \leq \frac{1}{n}\binom{z}{n-1} \\
&\quad\Longleftrightarrow \quad \frac{1}{n+1}\binom{x}{n}\frac{dx}{dz} - \frac{1}{n}\binom{z}{n-1} \leq 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{d}{dz}\left(\binom{x}{n+1} - \binom{y}{n+1} - \binom{z}{n}\right) \\
&\quad= \frac{d}{dz}\left(\frac{x-n}{n+1}\binom{x}{n} - \frac{y-n}{n+1}\binom{y}{n} - \frac{z-(n-1)}{n}\binom{z}{n-1}\right) \\
&\quad= \frac{d}{dz}\frac{x-n}{n+1}\binom{x}{n} - \frac{d}{dz}\frac{y-n}{n+1}\binom{y}{n} - \frac{d}{dz}\frac{z-(n-1)}{n}\binom{z}{n-1} \\
&\quad= \frac{d}{dx}\frac{dx}{dz}\frac{x-n}{n+1}\binom{x}{n} - \frac{d}{dz}\frac{z-(n-1)}{n}\binom{z}{n-1} \\
&\quad= \frac{x-n}{n+1}\frac{d}{dx}\binom{x}{n}\frac{dx}{dz} + \frac{1}{n+1}\binom{x}{n}\frac{dx}{dz} - \frac{z-(n-1)}{n}\frac{d}{dz}\binom{z}{n-1} - \frac{1}{n}\binom{z}{n-1} \\
&\quad= \frac{x-n}{n+1}\frac{d}{dz}\binom{z}{n-1} + \frac{1}{n+1}\binom{x}{n}\frac{dx}{dz} - \frac{z-(n-1)}{n}\frac{d}{dz}\binom{z}{n-1} - \frac{1}{n}\binom{z}{n-1} \\
&\quad= \frac{1}{n+1}\binom{x}{n}\frac{dx}{dz} - \frac{1}{n}\binom{z}{n-1} + \left(\frac{x-n}{n+1} - \frac{z-n+1}{n}\right)\frac{d}{dz}\binom{z}{n-1} \\
&\quad\leq \left(\frac{x-n}{n+1} - \frac{z-n+1}{n}\right)\frac{d}{dz}\binom{z}{n-1} \\
&\quad\leq 0
\end{aligned}
$$

and hence the latter result. $\square$

L. Lovász, *Combinatorial Problems and Exercises* (1993), Problem 13.31(a); R. M. Redheffer, *AMM* **103** (1996), 62–64.

► **67.** [*M20*] We often need to know that binomial coefficients aren't too large. Prove the easy-to-remember upper bound

$$\binom{n}{k} \le \left(\frac{ne}{k}\right)^k, \qquad \text{when } n \ge k \ge 0.$$

Proposition. $\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$.

Proof. Let n and k be arbitrary integers such that $n \ge k \ge 0$. We must show that

$$\binom{n}{k} \le \left(\frac{ne}{k}\right)^k.$$

In the case that $k = 0$, if we adopt the convention that $\left(\frac{ne}{k}\right)^0 = 1$, then

$$\binom{n}{0} = 1 \le \left(\frac{ne}{k}\right)^0.$$

Otherwise, since clearly

$$n^{\underline{k}} \le n^k$$

and from exercise 1.2.5-24

$$\frac{k^k}{e^{k-1}} \le k! \quad \text{iff} \quad \frac{1}{k!} \le \frac{e^{k-1}}{k^k},$$

we have that

$$\begin{aligned}\binom{n}{k} &= \frac{n^{\underline{k}}}{k!}\\ &\le \frac{n^k}{k!}\\ &\le \frac{e^{k-1}}{k^k}n^k\\ &= \frac{1}{e}\frac{n^k e^k}{k^k}\\ &= \frac{1}{e}\left(\frac{ne}{k}\right)^k\\ &\le \left(\frac{ne}{k}\right)^k.\end{aligned}$$

Hence

$$\binom{n}{k} \le \left(\frac{ne}{k}\right)^k$$

for all integers $n \ge k \ge 0$ as we needed to show. □

68. [*M25*] (A. de Moivre.) Prove that, if n is a nonnegative integer,

$$\sum_k \binom{n}{k} p^k (1-p)^{n-k} |k - np| = 2\lceil np \rceil \binom{n}{\lceil np \rceil} p^{\lceil np \rceil} (1-p)^{n+1-\lceil np \rceil}.$$

Proposition. $\sum_k \binom{n}{k} p^k (1-p)^{n-k} |k - np| = 2\lceil np \rceil \binom{n}{\lceil np \rceil} p^{\lceil np \rceil} (1-p)^{n+1-\lceil np \rceil}$.

Proof. Let n be a nonnegative integer. We must show that

$$\sum_k \binom{n}{k} p^k (1-p)^{n-k} |k - np| = 2\lceil np \rceil \binom{n}{\lceil np \rceil} p^{\lceil np \rceil} (1-p)^{n+1-\lceil np \rceil}.$$

But

$$\begin{aligned}
&\sum_k \binom{n}{k} p^k (1-p)^{n-k} |k - np| \\
&\quad = \sum_{k < \lceil np \rceil} \binom{n}{k} p^k (1-p)^{n-k} |k - np| \\
&\qquad + \sum_{\lceil np \rceil \le k} \binom{n}{k} p^k (1-p)^{n-k} |k - np| \\
&\quad = \sum_{k < \lceil np \rceil} \binom{n}{k} p^k (1-p)^{n-k} (np - k) \\
&\qquad + \sum_{\lceil np \rceil \le k} \binom{n}{k} p^k (1-p)^{n-k} (k - np) \\
&\quad = \sum_{k < \lceil np \rceil} \binom{n}{k} p^k (1-p)^{n-k} \left((k+1) \frac{n-k}{k+1} p - k(1-p) \right) \\
&\qquad + \sum_{\lceil np \rceil \le k} \binom{n}{k} p^k (1-p)^{n-k} \left(k(1-p) - (k+1) \frac{n-k}{k+1} p \right) \\
&\quad = \sum_{k < \lceil np \rceil} \left((k+1) \binom{n}{k+1} p^{k+1} (1-p)^{n+1-(k+1)} - k \binom{n}{k} p^k (1-p)^{n+1-k} \right) \\
&\qquad + \sum_{\lceil np \rceil \le k} \left(k \binom{n}{k} p^k (1-p)^{n+1-k} - (k+1) \binom{n}{k+1} p^{k+1} (1-p)^{n+1-(k+1)} \right) \\
&\quad = \lceil np \rceil \binom{n}{\lceil np \rceil} p^{\lceil np \rceil} (1-p)^{n+1-\lceil np \rceil} \\
&\qquad + \lceil np \rceil \binom{n}{\lceil np \rceil} p^{\lceil np \rceil} (1-p)^{n+1-\lceil np \rceil} \\
&\quad = 2\lceil np \rceil \binom{n}{\lceil np \rceil} p^{\lceil np \rceil} (1-p)^{n+1-\lceil np \rceil}
\end{aligned}$$

as we needed to show. $\square$

De Moivre, *Miscellanea Analytica* (1730), 101; H. Poincaré, *Calcul des Probabilités* (1896), 56–60; P. Diaconis and S. Zabell, *Statistical Science* **6** (1991), 284–302.